Rigid Analytic Quantum Groups

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface and specified in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the preface and specified in the text.

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Abstract

Following constructions in rigid analytic geometry, we introduce a theory of p-adic analytic quantum groups. We first define Fréchet completions $\widehat{U_q(\mathfrak{g})}$ and $\widehat{\mathcal{O}_q(G)}$ of the quantized enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$ and the quantized coordinate ring of the corresponding semisimple algebraic group G respectively. We consider these to be quantum analogues of the Arens-Michael envelope of the enveloping algebra $U(\mathfrak{g})$ and of the algebra of rigid analytic functions on the rigid analytification of G respectively. We show that these algebras are topological Hopf algebras and, by adapting techniques extracted from work of Ardakov-Wadsley, Schmidt and Emerton in p-adic representation theory, we also show that they are Fréchet-Stein algebras and use this to investigate an analogue of category $\mathcal{O}$ for $\widehat{U_q(\mathfrak{g})}$.

We then introduce a p-adic analytic analogue of Backelin and Kremnizer's construction of the quantum flag variety of a semisimple algebraic group, using a Banach completion of $\widehat{\mathcal{O}_q(G)}$. Our main result is a Beilinson-Bernstein localisation theorem in this context. We define a category of λ-twisted D-modules on this analytic quantum flag variety. This category has a distinguished object $\widehat{\mathcal{D}_q^\lambda}$ which plays the role of the sheaf of λ-twisted differential operators. We show that when λ is regular and dominant, the global section functor gives an equivalence of categories between the coherent λ-twisted D-modules and the finitely presented modules over the global sections of $\widehat{\mathcal{D}_q^\lambda}$.

The construction of this analytic quantum flag variety involves working with Banach comodules over the Banach completion $\widehat{\mathcal{O}_q(B)}$ of the quantum coordinate algebra of the Borel. Along the way, we also show that Banach comodules over $\widehat{\mathcal{O}_q(B)}$ can be naturally identified with what we call topologically integrable modules over the Banach completion of Lusztig's integral form of the quantum Borel.

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Chapter 1

Introduction

1.1 Background and motivation

Let L be a complete discrete valuation field of mixed characteristic $(0, p)$, with discrete valuation ring R and uniformizer π. Schneider and Teitelbaum introduced and studied locally analytic L-representations of a compact p-adic group G in a series of papers including [76, 74, 75]. These representations arise in several important places in Number Theory, most notably in the p-adic local Langlands program [29, 22, 35]. Many of these representations can be understood through the representation theory of the distribution algebra $D(G, L)$. More specifically, $D(G, L)$ is a *Fréchet-Stein algebra*, in the sense of [74], and such algebras have a well-behaved abelian category of *coadmissible modules*. These are expected to be the representations of interest. Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{g}_L$ denote its extension of scalars to L. Inside the distribution algebra lies the so-called *Arens-Michael envelope* $\widehat{U(\mathfrak{g}_L)}$ of the enveloping algebra of $\mathfrak{g}_L$, which is the subalgebra of distributions supported at the identity. It is also a Fréchet-Stein algebra and its coadmissible modules, which were first studied in [70, 72], can be thought of as a first approximation to the locally analytic representation theory of G.

In order to better understand the coadmissible $\widehat{U(\mathfrak{g}_L)}$-modules, Ardakov and Wadsley have recently started an ongoing program aiming to develop p-adic analytic analogues of D-modules, see [6, 8, 7, 5]. Their aim is to apply locally analytic Beilinson-Bernstein localisation results to these representations. In order to explain this better, we first quickly recall the classical Beilinson-Bernstein localisation:

Theorem 1.1.1 ([15]). *Let G be a reductive algebraic group over an algebraically closed field of characteristic 0, with Borel subgroup B and Lie algebra $\mathfrak{g}$. Let λ be a regular dominant weight, $X = G/B$ the flag variety of G and $\mathcal{D}_X^\lambda$ the sheaf of λ-twisted differential operators on X. Then the global section functor $\mathcal{M} \mapsto \mathcal{M}(X)$ gives an equivalence of categories*

$$\Gamma : \mathcal{D}_X^\lambda\text{-}mod \to U(\mathfrak{g})_\lambda\text{-}mod$$

between the category of (quasi-coherent)sheaves of $\mathcal{D}_X^\lambda$-modules and the category of $U(\mathfrak{g})$-modules with central character corresponding to λ.

The quasi-inverse of global sections in this equivalence is given by the *localisation* functor $\mathrm{Loc} : M \mapsto \mathcal{D}_X^\lambda \otimes_{U(\mathfrak{g})_\lambda} M$. Therefore one can localise in that way a representation of the Lie algebra $\mathfrak{g}$ to a sheaf on the flag variety. This allows in some sense to study $U(\mathfrak{g})$-modules locally, in the same way that one can study a module over a commutative ring A locally by considering the corresponding quasi-coherent sheaf of modules over Spec A. In fact, for this reason, we sometimes say that the above result means that the flag variety is *D-affine*. These localisation techniques were then used independently by Beilinson-Bernstein [15] and Brylinski-Kashiwara [26] to prove the Kazhdan-Lusztig conjectures [52]. More generally, the Beilinson-Bernstein theorem is viewed as the starting point of modern geometric representation theory, allowing for an interplay between techniques in representation theory and techniques in algebraic geometry.

Classically, $U(\mathfrak{g})$ may be thought of as an algebraic quantization of $\mathfrak{g}^*$. Returning to our p-adic setting, the algebra $\widehat{U(\mathfrak{g}_L)}$ is similarly in bijection with the ring $\mathcal{O}(\mathfrak{g}^{*,\mathrm{rig}})$ of rigid analytic functions on the rigid analytification $\mathfrak{g}^{*,\mathrm{rig}}$ of the affine variety $\mathfrak{g}^* = \mathrm{Spec}\ S(\mathfrak{g}_L)$, and it may be thought of as a 'rigid analytic quantization' of $\mathfrak{g}^{*,\mathrm{rig}}$. This makes it into a plausible candidate for a localisation theorem to hold, and indeed a Beilinson-Bernstein theorem for $\widehat{U(\mathfrak{g}_L)}$ was proved in [5], which in fact proved a more general equivariant result. It is hoped that it will have applications to the locally analytic representation theory of G. There have also been other approaches at using localisation techniques to understand locally analytic representations, notably by Schmidt [71] and Patel, Schmidt and Strauch [65, 66, 67].

We now return to the classical situation over the complex numbers. Around the same time as the early developments of geometric representation theory, the notion of quantum groups was introduced by Drinfel'd [33] and Jimbo [47]. Initially, these were invented in order to explain trigonometric $\mathcal{R}$-matrices in 2-dimensional solvable models in statistical mechanics, but since then they have proved to be an important tool in representation theory as well as in other areas. Roughly, these quantum groups are families of Hopf algebras, parametrized by an element q of the base field, which are in a suitable sense deformations of classical algebras. We are primarily interested in two families of such quantum algebras, namely the quantized enveloping algebra $U_q := U_q(\mathfrak{g})$ of a complex semisimple Lie algebra $\mathfrak{g}$ and the quantized coordinate algebra $\mathcal{O}_q := \mathcal{O}_q(G)$ of a complex (simply-connected) semisimple algebraic group G. The representation theory of U_q was first studied among others by Andersen, Polo and Wen [4], and Lusztig [57, 59, 58]. When the parameter q is not a root of unity, the representation theory of U_q is broadly analogous to the representation theory of $\mathfrak{g}$. When q is a root of unity, however, the representation theory of U_q is much more strongly related to, and has applications in, the representation theory of semisimple algebraic groups over fields of positive characteristics.

An important thread running through the theory of quantum groups has roots in the philosophy of Grothendieck: a space should be thought of as being the same thing as its category of (quasi-coherent) sheaves. Therefore one views the ring $\mathcal{O}_q$ as if it consisted of 'noncommutative functions' on a nonexistent object, namely a 'quantum group' corresponding to G, and $\mathcal{O}_q$-modules then correspond to quasi-coherent $\mathcal{O}_G$-modules under this analogy. But more generally, one can look for quantizations of general algebraic varieties, with the expectation that classical theorems on algebraic groups and their actions on varieties should have quantized analogues. Along with the development of noncommutative algebraic geometry by Artin-Zhang [9] and Rosenberg [68] among others, this naturally led to a definition of a quantum flag variety and of D-modules on it by Lunts-Rosenberg [56]. They conjectured that a Beilinson-Bernstein theorem should hold for their D-modules. This conjecture was essentially proved, after modifying a bit the notion of D-modules, by Backelin-Kremnizer [13] and Tanisaki [81] independently.

All of the above leads to the following question: can one quantize the algebras appearing in the study of locally analytic L-representations, and can one use Beilinson-Bernstein localisation techniques to then study the representation theory of these quantum analogues? While we believe this to be a natural question to ask, we also note that there has been some recent work developing what could be called noncommutative analytic geometry in [16]. In this light, we think that defining quantized analogues of rigid analytic spaces, for instance of analytic flag varieties, is an interesting endeavour in its own right.

The study of quantum groups in a p-adic analytic setting was first proposed by Soibelman in [80], where he introduced quantum deformations of the algebras of locally analytic functions on p-adic Lie groups and of the corresponding distribution algebra. His ideas were heavily influenced by the aforementioned work of Schneider and Teitelbaum. He conjectured among other things that his quantum distribution algebras are topological Hopf algebras and Fréchet-Stein algebras. Using a different approach than Soibelman, a quantum distribution algebra for GL_2 and a quantized p-adic upper half plane were defined in [83]. Recently, there has also been a new approach at constructing p-adic analytic quantum groups using Nichols algebra in [78]. However, besides these and a short note of Lyubinin [61], not much work has been done in this area. Nevertheless, we hope, perhaps even expect, that a successful theory of p-adic analytic quantum groups can be achieved in such a way that it will have applications to the representation theory of p-adic groups.

This thesis can be viewed as a contribution towards the development of such a theory. We will recall the precise definitions later, but we note here that the quantum groups $U_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$ depend purely on the combinatorics of the root system. Therefore these algebras may be defined over any field, and in particular over L. We then fix an element $q \in R^\times$ and assume that $q \equiv 1 \pmod{\pi}$ and that q is not a root of unity. Our goal then becomes to define suitable quantum analogues $\widehat{U_q}$ of the Arens-Michael envelope and $\widehat{\mathcal{O}_q}$ of the algebra of rigid analytic functions on the analytification G^{an} of G, and to obtain an analytic quantum Beilinson-Bernstein localisation theorem using these objects. While we are able to achieve the former, we do not quite achieve the latter. However we obtain an intermediate result, which we describe below.

Some of the work leading towards the Beilinson-Bernstein theorem for the Arens-Michael envelope is contained in the paper [6] of Ardakov and Wadsley, which itself proves a localisation theorem for certain completions of enveloping algebras. Let us briefly recall this result in more details. Let $\mathbf{G}$ be a simply connected split semisimple algebraic group over R with R-Lie algebra $\mathfrak{g}$ and let X be its flag scheme $\mathbf{G}/\mathbf{B}$. There is a family $(\widehat{\mathcal{U}_{n,L}})_{n\geq 0}$ of Banach completions of the envloping algebra $U(\mathfrak{g}_L)$ of the L-Lie algebra $\mathfrak{g}_L := \mathfrak{g} \otimes_R L$, which may be defined by taking the π-adic completion $\widehat{U(\pi^n\mathfrak{g})}$ and extending scalars to L. Moreover, for a weight λ, they defined a family $(\widehat{\mathcal{D}^{\lambda}_{n,L}})_{n\geq 0}$ of sheaves of completed deformed twisted crystalline differential operators on X. Their theorem then states:

Theorem 1.1.2 ([6])**.** *For any $n \geq 0$ and for λ regular and dominant, the global section functor gives an equivalence of categories between coherent sheaves of $\widehat{\mathcal{D}^{\lambda}_{n,L}}$-modules and finitely generated $\widehat{\mathcal{U}_{n,L}}$-modules with central character corresponding to λ.*

The connection between the algebras $\widehat{\mathcal{U}_{n,L}}$ and the Arens-Michael envelope is that $\widehat{U(\mathfrak{g}_L)}$ can be described as the projective limit $\varprojlim \widehat{\mathcal{U}_{n,L}}$, and a coadmissible $\widehat{U(\mathfrak{g}_L)}$-module is then given by a projective limit of finitely generated $\widehat{\mathcal{U}_{n,L}}$-modules with extra compatibility conditions. Hence we see that, in order to achieve a Beilinson-Bernstein theorem for $\widehat{U_q}$, one needs to first prove an analogue of Theorem 1.1.2 for quantum groups. That is what we aim to achieve in this thesis, in the case $n = 0$.

1.2 Quantum Arens-Michal envelopes and rigid analytic quantum groups

Our construction of $\widehat{U_q}$ and $\widehat{\mathcal{O}_q}$ is inspired by the theory developed by Ardakov and Wadsley in [8], where a general framework to show that certain algebras are Fréchet-Stein is also developed. The construction is inspired by the GAGA functor in rigid analytic geometry (see e.g. [20, Section 5.4]). Roughly, it goes as follows: the quantum group U_q contains an R-subalgebra U, which is known as the De Concini-Kac integral form. We then deform U, in the sense of [8], to obtain a family of R-subalgebras $(U_n)_{n\geq 0}$. We then set $\widehat{U_{q,n}} = \widehat{U_n}\otimes_R L$ for each $n \geq 0$, and $\widehat{U_q} = \varprojlim \widehat{U_{q,n}}$. The construction of $\widehat{\mathcal{O}_q}$ is similar. There is again an integral form $\mathcal{A}_q$ of $\mathcal{O}_q$. We may deform it and complete to obtain a family of Banach algebras $(\widehat{\mathcal{O}_{q,n}})_{n\geq 0}$, and we define $\widehat{\mathcal{O}_q} := \varprojlim \widehat{\mathcal{O}_{q,n}}$. Our results can then be summarized as follows:

Theorem A. *The algebras $\widehat{U_q}$ and $\widehat{\mathcal{O}_q}$ are Fréchet Hopf algebras. Moreover, they are Fréchet-Stein algebras.*

Here, a Fréchet Hopf algebra is a Fréchet algebra H equipped with continuous L-linear maps $\Delta : H \to H\widehat{\otimes}_L H$, $\varepsilon : H \to L$ and $S : H \to H$ satisfying the usual axioms for a Hopf algebra, where $H\widehat{\otimes}_L H$ denotes the completion of the tensor product $H \otimes_L H$ with respect to the projective tensor product topology.

Classically, the Arens-Michael envelope $\widehat{U(\mathfrak{g}_L)}$ can be identified as the completion of $U(\mathfrak{g}_L)$ with respect to all the submultiplicative semi-norms which extend the norm on L. We also show that our $\widehat{U_q}$ can be described similarly as the completion of U_q with respect to all the submultiplicative semi-norms on U_q which extend a canonical norm on the torus U_q^0 (see Chapter 2).

We note that there has been a succesful attempt at constructing a quantum Arens-Michael envelope for $\mathfrak{sl}_2$ and proving that it is a Fréchet-Stein algebra in [61], but the general case hasn't been tackled before. Although the object we construct is the same as theirs for $\mathfrak{sl}_2$, our constructions and proofs are different.

Next, we use the Fréchet-Stein structure of $\widehat{U_q}$ to define an analogue of the BGG category $\mathcal{O}$ for it. Indeed, there is a category $\mathcal{O}$ for quantum groups, see [3], which is a quantum analogue of the sum of the integral blocks inside the classical BGG category. Moreover, there already exists an analogue of category $\mathcal{O}$ for Arens-Michael envelopes, see [72], whose definition generalises straightforwardly to our quantum setting. Roughly, the category consists of those coadmissible modules over $\widehat{U_q}$ whose weight spaces are finite dimensional and such that the weights are contained in finitely many cosets in the weight lattice. We denote this new category by $\hat{\mathcal{O}}$. Then we prove:

Theorem B. *The functor $M \mapsto \widehat{U_q} \otimes_{U_q} M$ is an equivalence of categories between category $\mathcal{O}$ for U_q and the category $\hat{\mathcal{O}}$.*

The non-quantum version of this result is the main result of [72], and our proof follows theirs almost identically.

1.3 Quantum flag varieties and quantum D-modules

We now start describing in more details the constructions that lead to our Beilinson-Bernstein theorem, however leaving the precise definitions to the main body this thesis. The proof of Theorem 1.1.2 relies on the classical localisation theorem of Beilinson-Bernstein, and similarly we will have to rely on the quantum group analogue of Backelin and Kremnizer [13][1]. We briefly recall their constructions. Let $\mathcal{O}_q(B)$ be the quotient Hopf algebra of $\mathcal{O}_q$ corresponding to a Borel subgroup of G. Backelin and Kremnizer then define the quantum flag variety to be the category $\mathcal{M}_{B_q}(G_q)$ of $\mathcal{O}_q(B)$-equivariant $\mathcal{O}_q$-modules. Specifically, an object of this category is an $\mathcal{O}_q$-module equipped with a right $\mathcal{O}_q(B)$-comodule structure such that $\mathcal{O}_q$-action map is a comodule homomorphism. In this language, the global section functor Γ is the functor of taking $\mathcal{O}_q(B)$-coinvariants. They then define the ring of quantum differential operators on G to be the smash product algebra $\mathcal{D}_q = \mathcal{O}_q \# U_q$, and a λ-twisted D-module becomes an object M of the quantum flag variety equipped with an additional $\mathcal{D}_q$-action such that the $\mathcal{O}_q(B)$-coaction and the action of the quantum Borel subalgebra $U_q^{\geq 0} \subset U_q \subset \mathcal{D}_q$ 'differ by λ' (here λ is an element of the character group T_P of the weight lattice). There is also a distinguished object $\mathcal{D}_q^\lambda$ which represents global sections in the category of λ-twisted D-modules. Their main theorem is that, when λ is regular and dominant, the global section functor gives an equivalence of categories between λ-twisted D-modules and modules over $\Gamma(\mathcal{D}_q^\lambda)$. Moreover they show that the latter ring is isomorphic to $U_q^\lambda := U_q^{\mathrm{fin}}/J_\lambda$. Here U_q^{fin} corresponds to the finite part of U_q with respect to the adjoint action, and J_λ is the kernel of the central character corresponding to λ.

Nothing stops us from making completely analogous definitions using certain Banach completions $\widehat{\mathcal{O}_q}$, $\widehat{\mathcal{O}_q(B)}$ and $\widehat{\mathcal{D}_q}$ of these algebras (see section 1.4 below). That allows

[1] We point out here that there may be gaps with the Backelin-Kremnizer computation of global sections [13, Proposition 4.8], see [82, Remark 5.4]. This, however, does not stop D-affinity of the quantum flag variety from holding and thus the results of this thesis are not affected by it, with the exception of Theorem 5.4.18 which assumes that [13, Proposition 4.8] holds.

us to define what we call the analytic quantum flag variety as the category $\widehat{\mathcal{M}_{B_q}(G_q)}$ of $\widehat{\mathcal{O}_q(B)}$-equivariant Banach $\widehat{\mathcal{O}_q}$-modules, meaning that the objects of this category are Banach $\widehat{\mathcal{O}_q}$-modules which are also Banach $\widehat{\mathcal{O}_q(B)}$-comodules such that the $\widehat{\mathcal{O}_q}$-action map is a comodule homomorphism. We note that this category is not abelian. Instead it fits into Schneiders' framework of quasi-abelian categories [77]. In particular it has a derived category and, under suitable conditions, we can right derive left exact functors. The global section functor Γ here is also the functor of taking $\widehat{\mathcal{O}_q(B)}$-coinvariants, and we use this framework of quasi-abelian categories to make sense of the cohomology of Γ. We can then define λ-twisted D-modules to be objects in $\widehat{\mathcal{M}_{B_q}(G_q)}$ which are equipped with an additional $\widehat{\mathcal{D}_q}$-action such that the $\widehat{\mathcal{O}_q(B)}$-coaction and the action of $\widehat{U_q^{\geq 0}}$ differ by λ. There is also a distinguished object $\widehat{\mathcal{D}_q^\lambda}$ which represents global sections.

1.4 General strategy

Let us briefly outline the argument used by Ardakov and Wadsley in [6] to prove that one gets an equivalence of categories in Theorem 1.1.2. We will employ essentially the same strategy.

(i) They first work with integral versions of classical algebraic D-modules and show that large enough twists of coherent D-modules are acyclic and generated by their global sections. Using this, they then show that the category of coherent $\widehat{\mathcal{D}^\lambda_{n,L}}$-modules has a family of generators obtained from taking certain twists of $\widehat{\mathcal{D}^\lambda_{n,L}}$. In particular those are π-adic completions of algebraic D-modules.

(ii) The first step essentially reduces the problem to working with those coherent $\widehat{\mathcal{D}^\lambda_{n,L}}$-modules which can be 'uncompleted'. They then show that these are generated by their global sections. This uses the classical Beilinson-Bernstein theorem.

(iii) Finally, they show that completions of acyclic coherent D-modules are also acyclic. This uses technical facts from EGA [40] about the cohomology of a projective limit of sheaves.

(iv) Once you know that coherent $\widehat{\mathcal{D}^\lambda_{n,L}}$-modules are acyclic and generated by their global sections, the result follows from standard general facts.

In order to adapt this, we are first required to work with integral forms of quantum groups and the corresponding integral quantum flag variety (see Chapter 4). Specifically, we take the integral form $\mathcal{A}_q$ of $\mathcal{O}_q$ and let $\mathcal{B}_q$ be its image in the quotient Hopf algebra $\mathcal{O}_q(B)$. We are then able to define the category $\mathscr{C}_R$ of $\mathcal{B}_q$-equivariant $\mathcal{A}_q$-modules. We can also define an integral form $\mathcal{D}$ of the ring $\mathcal{D}_q$, and use it to define λ-twisted D-modules in $\mathscr{C}_R$ (here λ is an element of T_P^R, the character group over R of the weight lattice). These integral forms allow us to define the Banach completions we mentioned above by simply setting $\widehat{\mathcal{O}_q} := \widehat{\mathcal{A}_q} \otimes_R L$, $\widehat{\mathcal{O}_q(B)} := \widehat{\mathcal{B}_q} \otimes_R L$ and $\widehat{\mathcal{D}_q} := \widehat{\mathcal{D}} \otimes_R L$ respectively.

Unlike in the first step above, we are not able to show that large enough twists of coherent $\mathcal{D}$-modules are acyclic and generated by global sections, but we manage to show it for those which are annihilated by π. This turns out to be enough for the first two steps to work. Most of our work is then spent developing the correct tools from noncommutative algebraic geometry in the category $\widehat{\mathcal{M}_{B_q}(G_q)}$ in order for the ideas used in the third step to even make sense.

To have a version of step (iii) above, we need to work with the right sort of complexes, computing the cohomology of global sections, in order to apply the argument using EGA. To do so, it is convenient to work with proj categories. Indeed, the classical flag variety is isomorphic to $\mathrm{Proj}(\mathcal{O}(G/N))$, and Backelin-Kremnizer showed that $\mathcal{M}_{B_q}(G_q)$ is equivalent to $\mathrm{Proj}(\mathcal{O}_q(G/N))$ in the sense of Artin-Zhang [9]. We show that the integral quantum flag variety enjoys the same property. To obtain this result, one problem we ran into is that, while it is well-known that the algebra $\mathcal{O}_q$ is Noetherian, it isn't known in general whether its integral form $\mathcal{A}_q$ is also Noetherian (in type A, it is known to be true from Polo's appendix in [4]). That makes it non-trivial to define the objects which should play the role of coherent modules. Thankfully, we were able to prove that the integral form of $\mathcal{O}_q(G/N)$ is Noetherian, and using this we showed that the Noetherian objects in $\mathscr{C}_R$ are precisely those which are finitely generated over $\mathcal{A}_q$. Once this obstacle is cleared, the proof that we have a noncommutative projective scheme is essentially identical to the one in [13].

This result is essential because it allows us to define our promised complex which computes the cohomology of global sections for these integral forms. We think of this as a Čech-like complex. Using the Proj description of $\mathscr{C}_R$, one can in a suitable sense cover the category with analogues of the Weyl group translates of the big cell. The complexes are then obtained using general constructions from Rosenberg [68]. After taking π-adic completions, the objects of $\mathscr{C}_R$ are then naturally sent to another intermediate category, which we unoriginally call $\widehat{\mathscr{C}_R}$ and which is in some sense an integral form of $\widehat{\mathcal{M}_{B_q}(G_q)}$. We use the Weyl group localisations mentioned above to write down an analogue of our Čech-like complexes in this new integral category. After extending scalars, this gives us a Čech-like complex in the category $\widehat{\mathcal{M}_{B_q}(G_q)}$. This is the right object in order to apply the arguments from step (iii).

1.5 Main results

We now describe the main results of this thesis, which can be found in Chapter 5. Part of the definition of $\widehat{\mathcal{M}_{B_q}(G_q)}$ involves Banach comodules over $\widehat{\mathcal{O}_q(B)}$. We first give a more explicit description of these objects. We begin by defining what we call topologically integrable modules over a certain completion $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$ of $U_q^{\geq 0}$. Roughly, these are modules where the torus acts topologically semisimply and the positive part acts locally topologically nilpotently. The definition is inspired from work of Féaux de Lacroix [37], who developed a notion of semisimplicity for topological Fréchet modules. Our first main result is then:

Theorem C. *The category* $\mathbf{Comod}(\widehat{\mathcal{O}_q(B)})$ *of Banach right* $\widehat{\mathcal{O}_q(B)}$*-comodules is canonically equivalent to the category of topologically integrable* $\widehat{U^{res}(\mathfrak{b})}_L$*-modules.*

This result allows for a more intuitive understanding of what these comodules are, and also draws further parallels between our constructions and standard notions that appear in p-adic representation theory. We note that Banach comodules over a Banach coalgebra have also been studied in a more general, categorical setting in [54].

Our next result is that the cohomology of Γ in $\widehat{\mathcal{M}_{B_q}(G_q)}$ can be computed using the Čech-like complexes described above:

Theorem D. *For any $\mathcal{M} \in \widehat{\mathcal{M}_{B_q}(G_q)}$, the standard complex $\check{C}(\mathcal{M})$ computes $R\Gamma(\mathcal{M})$.*

We note that as a consequence of this, we obtain that Γ has finite cohomological dimension (something which wasn't obvious beforehand!). Both of these are essential in order to obtain a Beilinson-Bernstein theorem, but we also think of them as interesting results in their own right. We view our analytic quantum flag variety as being in some sense a noncommutative analytic space, and these results make it feasible to work with it.

Finally, with all the above at hand, we are able to run the strategy fom section 1.4 to obtain our version of Beilinson-Bernstein localisation. We call a D-module in $\widehat{\mathcal{M}_{B_q}(G_q)}$ coherent if it is finitely generated over $\widehat{\mathcal{D}_q}$.

Theorem E. *Suppose $\lambda \in T_P^R$ is regular and dominant. Then the functor Γ of global sections and the localisation functor Loc_λ are quasi-inverse equivalences of categories between the category $\mathrm{coh}(\widehat{\mathcal{D}^\lambda_{B_q}(G_q)})$ of λ-twisted coherent $\widehat{\mathcal{D}_q}$-modules on the analytic quantum flag variety and the category of finitely presented modules over $D := \Gamma(\widehat{\mathcal{D}^\lambda_q})$.*

The localisation functor Loc_λ here is defined by $M \mapsto \widehat{\mathcal{D}^\lambda_q} \otimes_D M$. Therefore we may think of this Theorem as saying that the category $\widehat{\mathcal{M}_{B_q}(G_q)}$ is D-affine. This result also implies that $\Gamma(\widehat{\mathcal{D}^\lambda_q})$ is a left coherent ring.

We are at the moment unable to compute the global sections $\Gamma(\widehat{\mathcal{D}^\lambda_q})$. Similarly to the situation with $\mathcal{O}_q$ and $\mathcal{A}_q$, while it is known that U_q^{fin} is Noetherian (see [48]), it doesn't appear to be known whether its integral form is as well. Under the hypothesis that it is Noetherian, we are able to prove that $\Gamma(\widehat{\mathcal{D}^\lambda_q}) \cong \widehat{U^\lambda_q}$ where the latter ring is a corresponding Banach completion of U_q^λ.

1.6 Structure of the thesis

We now describe the content of each chapter in turn. In Chapter 2, we summarize all the necessary facts the we will need about Hopf algebras, quantum groups, nonarchimedean functional analysis and quasi-abelian categories. We also included some general facts about R-algebras and modules. Note that the literature on Hopf algebras is usually purely written working over fields, whereas we provide a treatment over arbitrary commutative rings where possible. This higher level of generality is necessary due to the fact that we work with Hopf algebras defined over the ring R at several stages of this thesis. We also give an explicit description of the categories of $\mathcal{A}_q$-comodules and $\mathcal{B}_q$-comodules as the categories of integrable modules over Lusztig's integral forms of U_q and $U_q^{\geq 0}$ respectively. We believe this to be well-known, but we could not find any suitable reference for this, so we included a proof. This needed some general facts about duality for R-Hopf algebras which we also included with proofs.

In Chapter 3, we introduce the algebras $\widehat{U_q}$ and $\widehat{\mathcal{O}_q}$, and prove Theorem A. The Hopf algebra properties are easily deduced from general facts from nonarchimedean functional analysis. The main bulk of this chapter is then spent proving that these algebras are Fréchet-Stein. This is done mostly by adapting methods from Ardakov-Wadsley [8] and Emerton [36], who established certain general criteria for proving that some algebras are Fréchet-Stein. We then introduce the general formalism of topologically semisimple modules over the torus. This is inspired by the aforementioned work of Féaux de Lacroix [37]. Using this, we develop the basics of the category $\hat{\mathcal{O}}$: we introduce highest weight modules and Verma modules, and we then prove Theorem B. We also make a conjecture about an extension of the Harish-Chandra isomorphism to $\widehat{U_q}$.

In Chapter 4, we recall all the main definitions and constructions from [13]. We also include a proof that $\mathcal{D}_q$ is Noetherian. We then introduce the integral analogue $\mathscr{C}_R$ of the quantum flag variety and show that it is a Proj category. In doing so, we make heavy use of results about the cohomology of the induction functor for quantum groups from Andersen, Polo and Wen [4]. Using this, we obtain a Čech complex for computing the cohomology of global sections on $\mathscr{C}_R$. Finally, we introduce the λ-twisted D-modules on $\mathscr{C}_R$ and, using a result of Andersen and Jantzen [2], we prove that if a coherent D-modules is annihilated by π, then large enough twists of it are acyclic and generated by their global sections.

In Chapter 5, we introduce topologically integrable modules over the Banach completion $\widehat{U^{\mathrm{res}}(\mathfrak{b})_L}$ of Lusztig's integral form $U^{\mathrm{res}}(\mathfrak{b})$ for $U_q^{\geq 0}$, and show that these are equivalent to Banach $\widehat{\mathcal{O}_q(B)}$-comodules. The main idea here is somewhat inspired by steps (i)-(iii) earlier. Indeed, we reduce the problem to those modules that can be 'uncompleted', where the result will then follow from the fact that $\mathcal{B}_q$-comodules are equivalent to integrable $U^{\mathrm{res}}(\mathfrak{b})$-modules. In order to reduce to that case, we use the key fact that any Banach $\widehat{\mathcal{O}_q(B)}$-comodule $\mathcal{M}$ embeds topologically into $\mathcal{M}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$, equipped with the comodule structure $\mathrm{id}_{\mathcal{M}}\,\widehat{\otimes}\widehat{\Delta}$.

Next, we introduce the categories $\widehat{\mathscr{C}_R}$ and $\widehat{\mathcal{M}_{B_q}(G_q)}$. We then construct a Čech-like complex and prove that it computes the cohomology of global sections. The main technical tool we need here is some flatness results for completed tensor products from [19]. The theorem then follows essentially by using the fact that it holds for lattices modulo π^n for every n. Finally, we put everything together to prove our Beilinson-Bernstein theorem. The arguments here are essentially those from [6], with some small adjustments.

1.7 Conventions, notation and some elementary facts

The convention in locally analytic representation theory is to denote the base field by K. However it is also conventional to denote certain elements in the quantum group U_q by K, so we will use L to denote our base field. It will always denote a discrete valuation field of characteristic 0, with valuation ring R, uniformizer π and residue characteristic $p > 0$.

Unless explicitly stated otherwise, the term 'module' will be used to mean *left* module, and dually the term 'comodule' will be used to mean *right* comodule. We will say that a ring is Noetherian when it is both left and right Noetherian. All of our filtrations on modules or algebras will be positive and exhaustive unless specified otherwise.

In sections 2.1-2.3, k will denote an arbitrary field. From section 2.4 onwards, k will denote the residue field $R/\pi R$. Given an R-module M, we then denote by $M_k := M \otimes_R k$ its reduction modulo π and we will also write $M_L := M \otimes_R L$. Moreover we denote by $\widehat{M}$ its π-adic completion $\varprojlim M/\pi^n M$ and write $\widehat{M_L} := \widehat{M} \otimes_R L$.

Throughout this thesis, we will frequently make use of the following elementary facts about the ring R:

- R is a PID (see [73, Lemma 1.5]) and therefore an R-module is flat if and only if it is torsion-free (see [34, Corollary 6.3]). Moreover, if an R-module M is flat then so is its π-adic completion $\widehat{M}$ (see [11, Lemma 15.27.5]); and
- for any R-module M, the kernel of the map $M \to M \otimes_R L$ is the torsion submodule $\{m \in M : \pi^n m = 0 \text{ for some } n \geq 0\}$ (see [18, Lemma 2.10]).

Chapter 2

Background material

We begin by recalling some elementary facts concerning Hopf algebras, quantum groups, non-archimedean functional analysis and quasi-abelian categories.

2.1 Hopf algebras

We first collect standard facts about Hopf algebras. Suitable references for a more detailed treatment are e.g. [1, 23]. Note however that most of the treatment in the literature is concerned only with Hopf algebras over a field. We will consider more generally Hopf algebras over commutative rings, and later on more specifically over the ring R. We will therefore give references for some facts only when the proofs over fields are the same as over any commutative ring.

Throughout $\mathcal{R}$ will denote an arbitrary unital, commutative ring. We first recall the notion of an $\mathcal{R}$-algebra.

Definition 2.1.1. An $\mathcal{R}$-module A is called an $\mathcal{R}$-*algebra* if there exist $\mathcal{R}$-module maps $m : A \otimes_{\mathcal{R}} A \to A$ and $\imath : \mathcal{R} \to A$, called the *multiplication* and the *unit* respectively, such that the diagrams

$$\begin{array}{ccc} A \otimes_{\mathcal{R}} \mathcal{R} & \xrightarrow{\mathrm{id}\otimes\imath} & A \otimes_{\mathcal{R}} A \\ \cong\downarrow & & \downarrow m \\ A & \xrightarrow{\mathrm{id}} & A \end{array} \qquad \begin{array}{ccc} \mathcal{R} \otimes_{\mathcal{R}} A & \xrightarrow{\imath\otimes\mathrm{id}} & A \otimes_{\mathcal{R}} A \\ \cong\downarrow & & \downarrow m \\ A & \xrightarrow{\mathrm{id}} & A \end{array}$$

$$\begin{array}{ccc} A \otimes_{\mathcal{R}} A \otimes_{\mathcal{R}} A & \xrightarrow{m\otimes\mathrm{id}} & A \otimes_{\mathcal{R}} A \\ \mathrm{id}\otimes m\downarrow & & \downarrow m \\ A \otimes_{\mathcal{R}} A & \xrightarrow{m} & A \end{array}$$

all commute. The first two diagrams really just mean that A has a 1, while the last diagram is associativity. Concretely, the unit is just the map $\imath(\lambda) = \lambda.1_A$.

An $\mathcal{R}$-algebra A is *commutative* if the diagram

$$\begin{array}{ccc} A \otimes_{\mathcal{R}} A & \xrightarrow{\sigma} & A \otimes_{\mathcal{R}} A \\ m\downarrow & & \downarrow m \\ A & \xrightarrow{\mathrm{id}} & A \end{array}$$

commutes, where $\sigma(a \otimes b) = b \otimes a$ is the flip map.

An $\mathcal{R}$-module map $f : A \to B$ between two $\mathcal{R}$-algebras is called an $\mathcal{R}$-*algebra homomorphism* if the diagrams

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\imath_B} & B \\ \imath_A\downarrow & & \downarrow \mathrm{id} \\ A & \xrightarrow{f} & B \end{array} \qquad \begin{array}{ccc} A \otimes_{\mathcal{R}} A & \xrightarrow{f\otimes f} & B \otimes_{\mathcal{R}} B \\ m_A\downarrow & & \downarrow m_B \\ A & \xrightarrow{f} & B \end{array}$$

commute.

Having established this definition, we may now define the dual notion of a coalgebra.

Definition 2.1.2. An $\mathcal{R}$-module C is called an $\mathcal{R}$-*coalgebra* if there exist $\mathcal{R}$-module maps $\Delta : C \to C \otimes_{\mathcal{R}} C$ and $\varepsilon : C \to \mathcal{R}$, called the *comultiplication* and the *counit* respectively, such that the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\mathrm{id}} & C \\ \Delta\downarrow & & \downarrow \cong \\ C \otimes_{\mathcal{R}} C & \xrightarrow{\mathrm{id}\otimes\varepsilon} & C \otimes_{\mathcal{R}} \mathcal{R} \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\mathrm{id}} & C \\ \Delta\downarrow & & \downarrow \cong \\ C \otimes_{\mathcal{R}} C & \xrightarrow{\varepsilon\otimes\mathrm{id}} & \mathcal{R} \otimes_{\mathcal{R}} C \end{array}$$

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes_{\mathcal{R}} C \\ \Delta\downarrow & & \downarrow \Delta\otimes\mathrm{id} \\ C \otimes_{\mathcal{R}} C & \xrightarrow{\mathrm{id}\otimes\Delta} & C \otimes_{\mathcal{R}} C \otimes_{\mathcal{R}} C \end{array}$$

all commute. The last diagram is called the *coassociativity* property.

An $\mathcal{R}$-submodule $C' \subseteq C$ is called a *subcoalgebra* if the map $\Delta|_{C'}$ factors through $C' \otimes_{\mathcal{R}} C'$.

A coalgebra C is called *cocommutative* if the diagram

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes_{\mathcal{R}} C \\ \mathrm{id}\downarrow & & \downarrow \sigma \\ C & \xrightarrow{\Delta} & C \otimes_{\mathcal{R}} C \end{array}$$

commutes. In other words, $\Delta(C)$ is contained in the symmetric part of $C \otimes_{\mathcal{R}} C$.

An $\mathcal{R}$-module map $f : C \to D$ between two $\mathcal{R}$-coalgebras is called an $\mathcal{R}$-*coalgebra homomorphism* if the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\mathrm{id}} & C \\ f\downarrow & & \downarrow \varepsilon_C \\ D & \xrightarrow{\varepsilon_D} & \mathcal{R} \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{f} & D \\ \Delta_C\downarrow & & \downarrow \Delta_D \\ C \otimes C & \xrightarrow{f\otimes f} & D \otimes D \end{array}$$

commute.

Notation. We will use *Sweedler's notation* for the comultiplication. It goes as follows: for an element c of a coalgebra C, we will write $\Delta(c) = \sum c_1 \otimes c_2$[1]. Here c_1 and c_2 refer to variable elements of C, not uniquely determined. The subscripts refer to the positions of these elements in the tensor product expression for $\Delta(c)$. One can think of this as some sort of summation convention. For example, the counit axioms of a coalgebra can be written in the form $c = \sum \varepsilon(c_1)c_2 = \sum c_1\varepsilon(c_2)$. Moreover, by coassociativity of the comultiplication, we will write $\sum c_1 \otimes c_2 \otimes c_3$ to denote both $(\mathrm{id} \otimes \Delta)(\Delta(c))$ and $(\Delta \otimes \mathrm{id})(\Delta(c))$.

We now explain how to take quotients in coalgebras. Suppose that C is an $\mathcal{R}$-coalgebra and that $I \subseteq C$ is an $\mathcal{R}$-submodule. The next fact is well-known but we provide a proof for completeness:

Lemma 2.1.3. *The sequence*

$$(I \otimes_{\mathcal{R}} C) \oplus (C \otimes_{\mathcal{R}} I) \to C \otimes_{\mathcal{R}} C \to C/I \otimes_{\mathcal{R}} C/I \to 0$$

is exact.

Proof. Clearly the map $C \otimes_{\mathcal{R}} C \to C/I \otimes_{\mathcal{R}} C/I$ is surjective and its kernel K contains the image of $(I \otimes_{\mathcal{R}} C) \oplus (C \otimes_{\mathcal{R}} I)$ in $C \otimes_{\mathcal{R}} C$. For the reverse inclusion, by tensoring the short exact sequence

$$0 \to I \to C \to C/I \to 0$$

with I, C and C/I, we obtain three exact sequences. Part of these fit into a commutative diagram

$$\begin{tikzcd}
 & C \otimes_{\mathcal{R}} I \arrow[r] \arrow[d] & C/I \otimes_{\mathcal{R}} I \arrow[r] \arrow[d] & 0 \\
I \otimes_{\mathcal{R}} C \arrow[r] & C \otimes_{\mathcal{R}} C \arrow[r] \arrow[dr, "\theta"] & C/I \otimes_{\mathcal{R}} C \arrow[r] \arrow[d] & 0 \\
 & & C/I \otimes_{\mathcal{R}} C/I \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

where all the rows and columns are exact, and K is just the kernel of θ. Now a simple diagram chase gives us the reverse inclusion. □

Definition 2.1.4. A *coideal* in C is an $\mathcal{R}$-submodule I of C satisfying:

- $\varepsilon(I) = 0$; and
- $\Delta(I) \subseteq \mathrm{Im}\big((I \otimes_{\mathcal{R}} C) \oplus (C \otimes_{\mathcal{R}} I) \to C \otimes_{\mathcal{R}} C\big)$.

If I is a coideal in C, then by the above Lemma the quotient space A/I naturally has the structure of a coalgebra with the induced comultiplication and counit, and the quotient map $A \to A/I$ is a coalgebra homomorphism. More generally, the kernel of a coalgebra homomorphism is a coideal.

[1]some authors skip the summation symbol

Next, algebras and coalgebras share the useful property that they are closed under tensor products. If A and B are $\mathcal{R}$-algebras, define the multiplication in $A \otimes_{\mathcal{R}} B$ to be the natural one, i.e. defined by $(a_1 \otimes b_1).(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$, extended linearly. This can be expressed as

$$(A \otimes_{\mathcal{R}} B) \otimes_{\mathcal{R}} (A \otimes_{\mathcal{R}} B) \xrightarrow{\sigma_{23}} A \otimes_{\mathcal{R}} A \otimes_{\mathcal{R}} B \otimes_{\mathcal{R}} B \xrightarrow{m_A \otimes m_B} A \otimes_{\mathcal{R}} B$$

where σ_{23} is the map σ applied to the second and third factors. The unit of $A \otimes_{\mathcal{R}} B$ is given by the composite

$$\mathcal{R} \xrightarrow{\cong} \mathcal{R} \otimes_{\mathcal{R}} \mathcal{R} \xrightarrow{\imath_A \otimes \imath_B} A \otimes_{\mathcal{R}} B$$

Similarly, if C and D are $\mathcal{R}$-coalgebras, we can give a coalgebra structure to $C \otimes_{\mathcal{R}} D$ by defining the comultiplication to be

$$\Delta_{C \otimes_{\mathcal{R}} D}(c \otimes d) = \sum c_1 \otimes d_1 \otimes c_2 \otimes d_2.$$

This can be expressed as

$$C \otimes_{\mathcal{R}} D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes_{\mathcal{R}} C \otimes_{\mathcal{R}} D \otimes_{\mathcal{R}} D \xrightarrow{\sigma_{23}} C \otimes_{\mathcal{R}} D \otimes_{\mathcal{R}} C \otimes_{\mathcal{R}} D$$

where σ_{23} is as above. The counit is given by

$$\varepsilon_{C \otimes_{\mathcal{R}} D}(c \otimes d) = \varepsilon_C(c)\varepsilon_D(d).$$

We leave it to the reader to check that these satisfy the above definition of a coalgebra.

There is another construction that we can do to obtain new algebras/coalgebras. Given an algebra A, define $m_{\mathrm{op}} = m \circ \sigma$. Then, the triple $(A, m_{\mathrm{op}}, \imath)$ is an algebra, called the *opposite algebra*, denoted by A^{op}. Concretely, the multiplication in A^{op} is given by

$$a \cdot b = ba.$$

Dually, given a coalgebra C, we define $\Delta^{\mathrm{cop}} = \sigma \circ \Delta$. Then the triple $(C, \Delta^{\mathrm{cop}}, \varepsilon)$ is a coalgebra, called the *opposite coalgebra*, denoted by C^{cop}. It is clear from the definition that C is cocommutative if and only if $\Delta = \Delta^{\mathrm{cop}}$.

We now turn to bialgebras and Hopf algebras.

Definition 2.1.5. An $\mathcal{R}$-module B is an $\mathcal{R}$-*bialgebra* if it is both an $\mathcal{R}$-algebra and an $\mathcal{R}$-coalgebra such that the counit $\varepsilon : B \to \mathcal{R}$ and the comultiplication $\Delta : B \to B \otimes_{\mathcal{R}} B$ are $\mathcal{R}$-algebra homomorphisms. An $\mathcal{R}$-module map $f : A \to B$ between two $\mathcal{R}$-bialgebras is an $\mathcal{R}$-*bialgebra homomorphism* if it is both an algebra homomorphism and a coalgebra homomorphism.

Remark 2.1.6. Equivalently, an $\mathcal{R}$-module B is a bialgebra if it is both an algebra and a coalgebra and if the multiplication $m : B \otimes_{\mathcal{R}} B \to B$ and the unit $\imath : \mathcal{R} \to B$ are coalgebra homomorphisms, where we give $B \otimes_{\mathcal{R}} B$ the tensor coalgebra structure (see [1, Theorem

2.1.1]).

Definition 2.1.7. An $\mathcal{R}$-bialgebra $(H, m, \imath, \Delta, \varepsilon)$ is an *$\mathcal{R}$-Hopf algebra* if there exists an $\mathcal{R}$-module map $S : H \to H$ such that

$$\begin{tikzcd}
 & H \otimes_{\mathcal{R}} H \arrow[r, "S\otimes\mathrm{id}"] & H \otimes_{\mathcal{R}} H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rrr, "\imath\circ\varepsilon"] \arrow[dr, "\Delta"'] & & & H \\
 & H \otimes_{\mathcal{R}} H \arrow[r, "\mathrm{id}\otimes S"] & H \otimes_{\mathcal{R}} H \arrow[ur, "m"'] &
\end{tikzcd}$$

commutes, or in Sweedler's notation, such that

$$\varepsilon(h)1_H = \sum (S(h_1))h_2 = \sum h_1(S(h_2))$$

for all $h \in H$. The map S is called the *antipode* of H.

An *$\mathcal{R}$-Hopf algebra homomorphism* is an $\mathcal{R}$-bialgebra homorphism $f : H \to G$ between two $\mathcal{R}$-Hopf algebras H and G which preserves the antipode, i.e. such that $f \circ S_H = S_G \circ f$.

An $\mathcal{R}$-submodule I of an $\mathcal{R}$-Hopf algebra H is called a *Hopf ideal* if it is both a 2-sided ideal and a coideal, and if $S(I) \subseteq I$.

We now turn to duality issues. We will give a fuller picture of these questions when dealing with specific choices for $\mathcal{R}$, but for now we state a few general facts. Given an $\mathcal{R}$-module M, write M^* for its 'dual' $\mathrm{Hom}_{\mathcal{R}}(M, \mathcal{R})$. Similarly, given an $\mathcal{R}$-module map $\alpha : M \to N$, its dual $\alpha^* : N^* \to M^*$ is defined to be the map $\alpha^*(f) = f \circ \alpha$ for $f \in N^*$. Moreover, given two $\mathcal{R}$-modules M and N, there is a natural map

$$\phi_{M,N} : M^* \otimes_{\mathcal{R}} N^* \to (M \otimes_{\mathcal{R}} N)^*$$

given by $f \otimes g \mapsto (m \otimes n \mapsto f(m)g(n))$.

Lemma 2.1.8 ([23, Corollary 3.2.B])**.** *If C is a coalgebra then C^* is an algebra with multiplication $\Delta^* \circ \phi_{C,C}$ and unit ε^*. If C is cocommutative then A^* is commutative.*

We denote the product $\Delta^* \circ \phi_{C,C}(f \otimes g)$ of $f, g \in C^*$ by $f * g$. In Sweedler's notation, this is defined by

$$(f * g)(c) = \sum f(c_1)g(c_2)$$

for all $c \in C$. This product is called the *convolution* product. Note that when B is an $\mathcal{R}$-bialgebra, we can naturally extend the definition of the convolution product to $\mathrm{Hom}_{\mathcal{R}}(B, B)$ in the following way. For $f, g \in \mathrm{Hom}_{\mathcal{R}}(B, B)$ and $b \in B$, define

$$(f * g)(b) = \sum f(b_1)g(b_2).$$

Analogously to the Lemma, we then have that $(\mathrm{Hom}_{\mathcal{R}}(B, B), *, \imath \circ \varepsilon)$ is an $\mathcal{R}$-algebra (see [23, Proposition 3.2]). By definition, the antipode S of an $\mathcal{R}$-Hopf algebra H is then the inverse of the identity map with respect to the convolution product. Uniqueness of inverses then gives us that for any bialgebra B, if B has an antipode making it into a Hopf algebra, then that antipode is uniquely determined. The antipode satisfies further standard properties which we sum up here:

Lemma 2.1.9 ([1, Theorem 2.1.4])**.** *Let $(H, m, \imath, \Delta, \varepsilon, S)$ be an $\mathcal{R}$-Hopf algebra. Then $(H, m^{op}, \imath, \Delta^{cop}, \varepsilon, S)$ is also an $\mathcal{R}$-Hopf algebra, which we denote by H^{op}_{cop}. Moreover, $S : H \to H^{op}_{cop}$ is an Hopf algebra homomorphism. This says that*

$$S(xy) = S(y)S(x), \quad S(1) = 1$$

and

$$\Delta(S(x)) = \sum S(x_2) \otimes S(x_1), \quad \varepsilon(S(x)) = \varepsilon(x)$$

for all $x, y \in H$.

Example 2.1.10. We now give a few standard examples of Hopf algebras. Note first that since in a bialgebra, the counit and comultiplication are algebra homomorphisms, it's enough to define them on a set of algebra generators. Also, it's enough to check commutativity of the diagrams in the axioms for a Hopf algebra by applying the maps to a set of algebra generators.

(i) Let G be a group, and $\mathcal{R}G$ be its group algebra. Then $\mathcal{R}G$ is a Hopf algebra. The comultiplication is defined by $\Delta(g) = g \otimes g$ for all $g \in G$, the counit is given by $\varepsilon(g) = 1$ for all $g \in G$ and the antipode is given by $S(g) = g^{-1}$, all extended linearly. The axioms for a Hopf algebra are completely straightforward to check. The group algebra $\mathcal{R}G$ is cocommutative since Δ is symmetric, and is commutative if and only if the group G is abelian.

(ii) Let $\mathfrak{g}$ be a Lie algebra over $\mathcal{R}$. Then its universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra under

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad S(x) = -x, \quad \varepsilon(x) = 0$$

for $x \in \mathfrak{g}$, extended to $\mathcal{R}$-algebra maps on $U(\mathfrak{g})$. Again, $U(\mathfrak{g})$ is cocommutative and is commutative if and only if $\mathfrak{g}$ is abelian.

(iii) Let G be a linear algebraic group over a field k, and let $\mathcal{O}(G)$ be its algebra of regular functions. Consider the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$, which are both morphisms of varieties. These induce k-algebra homomorphisms

$$\begin{aligned}
\Delta : \mathcal{O}(G) &\longrightarrow \mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes \mathcal{O}(G), \\
f &\longmapsto f \circ m \\
\varepsilon : \mathcal{O}(G) &\longrightarrow k, \\
f &\longmapsto f(1) \\
\text{and} \quad S : \mathcal{O}(G) &\longrightarrow \mathcal{O}(G) \\
f &\longmapsto f \circ i
\end{aligned}$$

which make $\mathcal{O}(G)$ into a k-Hopf algebra. Indeed, the Hopf algebra axioms then simply become the group axioms for G. This time, $\mathcal{O}(G)$ is commutative, and is cocommutative if and only if G is abelian.

For example, when $G = \mathrm{GL}_n$, we have $\mathcal{O}(G) = k[x_{ij}, T]/(\det(x_{ij}) \cdot T - 1)$ as a k-algebra. The Hopf algebra structure is given by the duals of the usual matrix operations:

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \quad \varepsilon(x_{ij}) = \delta_{ij}, \quad S(x_{ij}) = (X^{-1})_{ij}$$

where $X = (x_{ij})$. Since $\det(X^{-1}) = \det(X)^{-1}$, we have $S(T) = \det(x_{ij}) = T^{-1}$.

Remark 2.1.11. In general, an element h in a Hopf algebra H is called *group-like* if it satisfies $\Delta(h) = h \otimes h$.

We will see later further examples of Hopf algebras when working with quantum groups. We now turn to the definition of modules and comodules.

Definition 2.1.12. A *left module* M over an $\mathcal{R}$-algebra A is an $\mathcal{R}$-module equipped with an $\mathcal{R}$-module homomorphism $\lambda : A \otimes_{\mathcal{R}} M \to M$ such that

$$\begin{array}{ccc}
A \otimes_{\mathcal{R}} A \otimes_{\mathcal{R}} M & \xrightarrow{\mathrm{id} \otimes \lambda} & A \otimes_{\mathcal{R}} M \\
{\scriptstyle m \otimes \mathrm{id}} \big\downarrow & & \big\downarrow {\scriptstyle \lambda} \\
A \otimes_{\mathcal{R}} M & \xrightarrow{\quad \lambda \quad} & M
\end{array}$$

and

$$\begin{array}{ccc}
\mathcal{R} \otimes_{\mathcal{R}} M & \xrightarrow{\imath \otimes \mathrm{id}} & A \otimes_{\mathcal{R}} M \\
& {\scriptstyle \cong} \searrow & \big\downarrow {\scriptstyle \lambda} \\
& & M
\end{array}$$

commute. An $\mathcal{R}$-module homomorphism $f : M \to N$ between two A-modules is an A-module homomorphism if

$$\begin{array}{ccc}
A \otimes_{\mathcal{R}} M & \xrightarrow{\mathrm{id} \otimes f} & A \otimes_{\mathcal{R}} N \\
{\scriptstyle \lambda_M} \big\downarrow & & \big\downarrow {\scriptstyle \lambda_N} \\
M & \xrightarrow{\quad f \quad} & N
\end{array}$$

commutes.

A *right comodule* M over an $\mathcal{R}$-coalgebra C is an $\mathcal{R}$-module equipped with an $\mathcal{R}$-module homomorphism $\rho : M \to M \otimes_{\mathcal{R}} C$ such that

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes_{\mathcal{R}} C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ M \otimes_{\mathcal{R}} C \arrow[r, "\rho\otimes\mathrm{id}"] & M \otimes_{\mathcal{R}} C \otimes_{\mathcal{R}} C \end{tikzcd}$$

and

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[dr, "\cong"'] & M \otimes_{\mathcal{R}} C \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\ & M \otimes_{\mathcal{R}} \mathcal{R} \end{tikzcd}$$

commute. We will refer to the map ρ as the *coaction*. An $\mathcal{R}$-submodule $N \subseteq M$ is called a *subcomodule* if the map $\rho|_N$ factors through $N \otimes_{\mathcal{R}} C$.

A *comodule homomorphism* $f : M \to N$ is an $\mathcal{R}$-module homomorphism such that

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho_M"'] & N \arrow[d, "\rho_N"] \\ M \otimes_{\mathcal{R}} C \arrow[r, "f\otimes\mathrm{id}"] & N \otimes_{\mathcal{R}} C \end{tikzcd}$$

commutes.

Notation. Sweedler's notation for comodules is completely analogous to the notation for coalgebras, i.e. we write $\rho(m) = \sum m_1 \otimes m_2$. Moreover, we will write $\sum m_1 \otimes m_2 \otimes m_3$ to denote both $(\mathrm{id} \otimes \Delta)(\rho(m))$ and $(\rho \otimes \mathrm{id})(\rho(m))$.

Example 2.1.13. (i) Any $\mathcal{R}$-algebra A is a module over itself by left multiplication. Dually any $\mathcal{R}$-coalgebra C is a right comodule over itself with coaction $\rho = \Delta$. More generally, if M is any $\mathcal{R}$-module, then $M \otimes_{\mathcal{R}} C$ is a right C-comodule with coaction $\mathrm{id} \otimes \Delta$. In both cases the comodule axioms are immediate from the coalgebra axioms.

(ii) Suppose that G is a linear algebraic group over a field k. Then the category of rational representations of G is canonically equivalent to the category of right $\mathcal{O}(G)$-comodules, see [46, Section I.2.8].

(iii) Let B be an $\mathcal{R}$-bialgebra and V an $\mathcal{R}$-module. The *trivial representation* of B on V is the B-module structure given by $b \cdot v = \epsilon(b)v$ for $b \in B$ and $v \in V$. More generally, if V is a left B-module, then $v \in V$ is said to be *invariant* under the action of B if $b \cdot v = \varepsilon(b)v$ for all $b \in B$. Similarly, the trivial right comodule structure on V is defined using the unit $\imath$, i.e. for $v \in V$, we set $\rho(v) = v \otimes 1$. More generally, if V is a right B-comodule, then $v \in V$ is said to be *coinvariant* if $\rho(v) = v \otimes 1$.

(iv) If H is an $\mathcal{R}$-Hopf algebra, the *adjoint representation* of H on itself is given by

$$\mathrm{ad}(h)(h') = \sum h_1 h' S(h_2).$$

Note that for $H = \mathcal{R}G$ a group algebra, this is just the conjugation action, and similarly for $H = U(\mathfrak{g})$, it is the usual adjoint representation of $\mathfrak{g}$.

(v) For H as above, if M is any H-module then M^* is also an H-module via the action

$$(h \cdot f)(m) = f(S(h)m)$$

for $h \in H$, $f \in M^*$ and $m \in M$.

Remark 2.1.14. The following observations will be useful at several points later on. Suppose that M is a right C-comodule for some $\mathcal{R}$-coalgebra C, with coaction $\rho_M : M \to M \otimes_{\mathcal{R}} C$. Note that by the axioms of comodules, the composite

$$(\mathrm{id}_M \otimes \varepsilon) \circ \rho_M = \mathrm{id}_M$$

so that the map ρ splits and M is a direct summand of $M \otimes_{\mathcal{R}} C$ as an R-module. Moreover, the map $\mathrm{id}_M \otimes \Delta$ makes $M \otimes_{\mathcal{R}} C$ into a C-comodule, so that the first diagram in the definition of a right comodule and the splitting say together that M identifies via ρ_M with a subcomodule of $M \otimes_{\mathcal{R}} C$.

Next we describe how to tensor modules/comodules together. Suppose that B is an $\mathcal{R}$-bialgebra and let M and N be two B-modules. Then we may give to $M \otimes_{\mathcal{R}} N$ the structure of a B-module as follows. It is clear that $M \otimes_{\mathcal{R}} N$ is a $B \otimes_{\mathcal{R}} B$-module via

$$(b \otimes c) \cdot (m \otimes n) = b \cdot m \otimes c \cdot n.$$

Thus we may view it as a B-module by letting $b \in B$ act on $M \otimes_{\mathcal{R}} N$ via $\Delta(b)$. This gives a module structure because Δ is an algebra homomorphism.

Now suppose that M and N are two right B-comodules. Then we may give to $M \otimes_{\mathcal{R}} N$ the structure of a comodule as follows: the coaction is set to be

$$M \otimes_{\mathcal{R}} N \xrightarrow{\rho_M \otimes \rho_N} M \otimes_{\mathcal{R}} B \otimes_{\mathcal{R}} N \otimes_{\mathcal{R}} B \xrightarrow{\sigma_{23}} (M \otimes_{\mathcal{R}} N) \otimes_{\mathcal{R}} B \otimes_{\mathcal{R}} B \xrightarrow{\mathrm{id} \otimes m} M \otimes_{\mathcal{R}} N \otimes_{\mathcal{R}} B$$

or in Sweedler's notation, $m \otimes n \mapsto \sum m_1 \otimes n_1 \otimes m_2 n_2$.

Definition 2.1.15. If B is as above, we say that an $\mathcal{R}$-algebra A is a *left B-module algebra* if it is a left B-module such that

(i) $b \cdot (xy) = \sum (b_1 \cdot x)(b_2 \cdot y)$ for $b \in B$ and $x, y \in A$,

(ii) $b \cdot 1_A = \varepsilon(b) 1_A$ for $b \in B$.

Equivalently, the multiplication and unit maps m_A and $\imath_A$ on A are B-module maps, where $A \otimes_{\mathcal{R}} A$ is made into a left B-module as above and $\mathcal{R}$ is viewed as the trivial B-module. We can then dualise this definition: a *right B-comodule algebra* is an $\mathcal{R}$-algebra A such that A is a right B-comodule and the maps m_A and $\imath_A$ are right B-comodule maps, where $A \otimes_{\mathcal{R}} A$ is made into a right B-comodule as above and $\mathcal{R}$ is viewed as the trivial B-comodule.

Example 2.1.16. (i) Any $\mathcal{R}$-Hopf algebra H is a left H-module algebra under the adjoint action.

(ii) If $B = \mathcal{R}G$ is the group algebra of a group G, then a B-module algebra is an algebra A on which the group G acts by algebra homomorphisms. Similarly, if $B = U(\mathfrak{g})$ then a B-module algebra is an algebra A on which the Lie algebra $\mathfrak{g}$ acts by derivations.

Definition 2.1.17. Let H be an $\mathcal{R}$-Hopf algebra, and let A be an H-module algebra. The *smash product algebra* $A\#H$ is defined to be $A \otimes_{\mathcal{R}} H$ as an $\mathcal{R}$-module, but with multiplication given by

$$(a \otimes u) \cdot (b \otimes v) = \sum au_1(b) \otimes u_2 v$$

for $a, b \in A$ and $u, v \in H$.

We make a couple of observations from this definition. First note that, by plugging $u = v = 1$ and $a = b = 1$ respectively in the formula for the product in $A\#H$, we obtain $(a \otimes 1) \cdot (b \otimes 1) = ab \otimes 1$ and $(1 \otimes u) \cdot (1 \otimes v) = 1 \otimes uv$ so that both A and H arise as subalgebras of $A\#H$. Moreover $(a \otimes 1) \cdot (1 \otimes u) = (a \otimes u)$. Thus we see that A and H generate $A\#H$. From now on we drop the tensor signs, and write au for $a \otimes u$.

Finally note that, inside $A\#H$, the action $u(a)$ of $u \in H$ on $a \in A$ coincides with the adjoint action $\sum u_1 a S(u_2)$. Indeed, for any $v \in H$ and $b \in A$, we have

$$vb = (1 \otimes v)(b \otimes 1) = \sum v_1(b) \otimes v_2 = \sum v_1(b) v_2.$$

Hence by plugging this in we get

$$\sum u_1 a S(u_2) = \sum u_1(a) u_2 S(u_3) = \sum \varepsilon(u_2) u_1(a) = u(a)$$

as claimed.

Example 2.1.18. Let G be a linear algebraic group over k, and let $\mathfrak{g} = \mathrm{Lie}(G)$. Then by definition, the Lie algebra $\mathfrak{g}$ acts by derivations on $\mathcal{O}(G)$. By Example 2.1.16(ii), this means $\mathcal{O}(G)$ is a left $U(\mathfrak{g})$-module algebra and hence we may form the smash product $\mathcal{O}(G)\#U(\mathfrak{g})$. Now let $\mathcal{D}_G$ denote the ring of global crystalline differential operators on G. Then by [39, Proposition 5.3] and by the above remarks, $\mathcal{D}_G$ is generated by $U(\mathfrak{g})$ and $\mathcal{O}(G)$ as an algebra and its defining relations are all holding in $\mathcal{O}(G)\#U(\mathfrak{g})$, hence there is a natural surjection $\mathcal{D}_G \to \mathcal{O}(G)\#U(\mathfrak{g})$. In fact, by [39, Proposition 5.2], $\mathcal{D}_G = \mathrm{Sym}_{\mathcal{O}(G)}(\mathfrak{g}) \cong \mathcal{O}(G) \otimes_k U(\mathfrak{g})$ as a k-vector space, hence we see that the previous surjection is an isomorphism.

2.2 Matrix coefficients and Hopf duals over fields

The dual of Lemma 2.1.8 is not true in general since it may not be true that the natural map $\varphi_A : A^* \otimes_{\mathcal{R}} A^* \to (A \otimes_{\mathcal{R}} A)^*$ is an isomorphism, where A is an $\mathcal{R}$-algebra. So the dual of the multiplication map on an $\mathcal{R}$-algebra A need not take all values in $A^* \otimes_{\mathcal{R}} A^*$.

Therefore we will need to be a bit more restrictive in our notion of a dual space. In this Section we will investigate this notion of dual over fields, and in the next Section over DVRs. All this material is well known over fields, but we think it is worthwhile to go through the details there in order for the theory over DVRs to be more intuitive. Hence we will provide some proofs. Of course we do not claim any originality there, all the results over fields can be found in e.g. [1, 23, 25]. We first begin in full generality, with an $\mathcal{R}$-algebra A.

Definition 2.2.1. Let M be an A-module, and let $f \in H^*$ and $v \in M$. The *matrix coefficient* $c^M_{f,v} \in H^*$ is defined by $c^M_{f,v}(x) := f(xv)$ for $x \in H$.

To justify the name, note that if for example M is free of finite rank over $\mathcal{R}$ with basis $v_1, \dots, v_n$ and corresponding dual basis $f_1, \dots, f_n$, then $c^M_{f_j,v_i}(x)$ is simply the (i,j)-th entry in the matrix for the action of x on M with respect to the above basis.

Suppose now that H is an $\mathcal{R}$-Hopf algebra. Then we know that H^* is an $\mathcal{R}$-algebra by Lemma 2.1.8, with multiplication Δ^* and ε is its 1. Hence it makes sense to consider the product of two matrix coefficients. What we need is summarized in the following:

Lemma 2.2.2 ([25, Lemma I.7.3])**.** *Let A, H be as above.*

(i) Suppose M and N are A-modules. Then

$$c^M_{f,m} + c^N_{g,m} = c^{M\oplus N}_{(f,g),(m,n)}$$

for all $m \in M$, $f \in M^*$, $n \in N$, $g \in N^*$.

(ii) Suppose that M is an A-module which is free of finite rank over $\mathcal{R}$, with basis $v_1, \dots, v_n$ and corresponding dual basis $f_1, \dots, f_n$, and suppose that $v \in M$ and $f \in M^*$. Then

$$m^*(c^M_{f,v}) = \sum_{i=1}^{n} \varphi_A(c^M_{f,v_i} \otimes c^M_{f_i,v})$$

where m is the multiplication map in A.

(iii) Suppose we're given two H-modules M and N, and elements $v \in M$, $f \in M^*$, $w \in N$ and $g \in N^*$. Then

$$c^M_{f,v} \cdot c^N_{g,w} = c^{M\otimes_{\mathcal{R}} N}_{f\otimes g, v\otimes w}, \quad S^*(c^M_{f,v}) = c^{M^*}_{v,f}$$

where S is the antipode of H.

Proof. Part (i) follows immediately from the definition. For (ii), a priori we only know that $m^*(c^M_{f,v}) \in (A \otimes_{\mathcal{R}} A)^*$. So pick $a, b \in A$, and suppose that $bv = \sum_{i=1}^n \lambda_i v_i$. Then by definition we have $\lambda_i = c^M_{f_i,v}(b)$ and so

$$\begin{aligned} m^*(c^M_{f,v})(a \otimes b) &= c^M_{f,v}(ab) \\ &= f(abv) \\ &= \sum_{i=1}^n \lambda_i f(av_i) \\ &= \sum_{i=1}^n c^M_{f_i,v}(b) c^M_{f,v_i}(a) \\ &= \sum_{i=1}^n \varphi_A(c^M_{f,v_i} \otimes c^M_{f_i,v})(a \otimes b) \end{aligned}$$

as required.

(iii) Pick $h \in H$. Then we have

$$\begin{aligned} (c^M_{f,v} \cdot c^N_{g,w})(h) &= \sum c^M_{f,v}(h_1) c^N_{g,w}(h_2) \\ &= \sum f(h_1 v) g(h_2 w) \\ &= (f \otimes g)(\Delta(h) \cdot (v \otimes w)) \\ &= c^{M \otimes_{\mathcal{R}} N}_{f \otimes g, v \otimes w}(h) \end{aligned}$$

by definition of the action on the tensor $M \otimes_{\mathcal{R}} N$. For the second part, we have

$$\begin{aligned} S^*(c^M_{f,v})(h) &= c^M_{f,v}(S(h)) \\ &= f(S(h)v) \\ &= (h \cdot f)(v) \\ &= c^{M^*}_{v,f}(h) \end{aligned}$$

by definition of the action on M^*. □

Hence we see that the matrix coefficients over a Hopf algebra are closed under taking products and under the action of S^* and m^*.

We now consider the case where $\mathcal{R} = k$ is some field. The map φ_A described before is now always injective. We can define our notion of dual as follows:

Definition 2.2.3. Suppose that A is a k-algebra. We set

$$A^\circ = \{f \in A^* : f(I) = 0 \text{ for some ideal } I \text{ of } A \text{ with } \dim_k(A/I) < \infty\}.$$

This is called *finite dual* or *Hopf dual* of A.

Lemma 2.2.4 ([25, I.7.2 & Exercise I.7.B])**.** *The set A° equals the set of matrix coefficients of finite dimensional A-modules. Hence it is a vector subspace of A^*.*

Proof. If M is a finite dimensional A-module, then for any $v \in M$ and $f \in M^*$, we have $c^M_{f,v}(I) = 0$ where $I = \mathrm{ann}_A(M) = \ker(A \to \mathrm{End}_k(M))$. The fact that I has finite codimension then follows from the fact that $\mathrm{End}_k(M)$ is finite dimensional.

Conversely, if $f \in A^\circ$, let I be an ideal of finite codimension killed by f. Then f factors through A/I and induces a map $\bar{f} : M \to k$ where $M := A/I$ is a finite dimensional A-module. Let v denote the image of 1 in the quotient A/I, and note that $\bar{f} \in M^*$ by definition. Given $a \in A$, write $\bar{a} = a + I \in A/I$. Then by definition we have

$$f(a) = \bar{f}(\bar{a}) = \bar{f}(a \cdot v) = c^M_{\bar{f},v}(a)$$

so that we get $f = c^M_{\bar{f},v}$.

The last part now follows immediately from Lemma 2.2.2(i). □

From this we get:

Corollary 2.2.5 ([1, Theorem 2.2.12]). *If A is an algebra with multiplication m and unit $\imath$, then A° is a coalgebra with $\Delta = m^*|_{A^\circ}$ and $\varepsilon = \imath^*|_{A^\circ}$. Furthermore, if A is commutative then A° is cocommutative.*

Proof. Note that ε is by definition a map $A^\circ \to k$, and more specifically it is the map $f \mapsto f(1)$. Since finite dimensional A-modules are free over k, it follows from Lemmas 2.2.2(ii) & 2.2.4 that $\Delta(A^\circ) \subseteq A^\circ \otimes_k A^\circ$. But now the coalgebra axioms are just a formal consequence of the algebra axioms on A. Indeed, if $\Delta(f) = \sum f_1 \otimes f_2$, then by definition of Δ this means that $f(ab) = \sum f_1(a) f_2(a)$ for all $a, b \in A$. So for instance,

$$\big((\mathrm{id} \otimes \varepsilon) \circ \Delta\big)(f)(a) = \sum f_1(a) f_2(1) = f(a \cdot 1) = f(a)$$

for every $a \in A$. Hence we see that $(\mathrm{id}_{A^\circ} \otimes \varepsilon) \circ \Delta = \mathrm{id}_{A^\circ}$ as required. Similarly we have $(\varepsilon \otimes \mathrm{id}_{A^\circ}) \circ \Delta = \mathrm{id}_{A^\circ}$. Finally, using the inclusion $A^\circ \otimes_k A^\circ \otimes_k A^\circ \subset (A \otimes_k A \otimes_k A)^*$, we get for every $a, b, c \in A$ and $f \in A^\circ$ that

$$\big((\Delta \otimes \mathrm{id}) \circ \Delta)\big)(f)(a \otimes b \otimes c) = \sum (f_1)_1(a)(f_1)_2(b) f_2(c) = \sum f_1(a \cdot b) f_2(c) = f\big((a \cdot b) \cdot c\big),$$

and a similar calculation yields

$$\big((\mathrm{id} \otimes \Delta) \circ \Delta)\big)(f)(a \otimes b \otimes c) = f\big(a \cdot (b \cdot c)\big).$$

Thus coassociativity follows from associativity. The cocommutativity statement is also clear. □

Now suppose that we are given a k-Hopf algebra H. Then by the above H° is a k-coalgebra. Moreover, since finite dimensional H-modules are closed under tensor products and duals, we see from Lemmas 2.2.2(iii) & 2.2.4 that H° is preserved by S^* and is closed under multiplication. Also $\varepsilon \in H^\circ$ since it's an algebra homomorphism $H \to k$ and hence its kernel is a 2-sided ideal of codimension 1. Thus we see that H° is a k-algebra. Putting everything together this gives:

Lemma 2.2.6 ([23, Theorem 3.6]). *Suppose that we are given a k-Hopf algebra H. Then H° is also a k-Hopf algebra with structure given by the duals of the Hopf algebra maps on H.*

We now turn to modules and comodules. Given a k-algebra A, an A-module M is called *locally finite* if for every $m \in M$, $\dim_k A \cdot m < \infty$.

Lemma 2.2.7 ([25, Lemma I.9.16])**.** *Let A be a k-algebra and C be a k-coalgebra.*

(i) If M is a right C-comodule, then M can be given canonically the structure of a left C^*-module by setting

$$f \cdot m = \sum f(m_2)m_1$$

for $f \in C^*$ and $m \in M$. This module structure is locally finite.

(ii) If M is a left A-module, then M can be given the structure of an A°-comodule whose associated left A-module (as above) is M if and only if M is locally finite.

2.3 Hopf duals over DVRs and their comodules

We now aim to make analogous constructions to the previous Section when $\mathcal{R} = R$. Most of our proofs work similarly as over fields, but one has to be a bit careful when dealing with torsion.

We begin by a couple of observations. Note that for any R-module M, its dual M^* is always torsion-free since R is a domain: if $\pi f = 0$ then $\pi f(u) = 0$ for all $u \in H$ and so $f(u) = 0$ for all u. Hence it follows that the canonical map $\varphi_M : M^* \otimes_R M^* \to (M \otimes_R M)^*$ is always injective since it becomes injective after extending scalars to L and $M^* \otimes_R M^*$ is torsion-free.

For the entirety of this Section, H will denote a fixed R-Hopf algebra. For our purposes, it will be enough to work in the case where H is *torsion-free.* First we wish to define a notion of Hopf dual. Since H has no torsion, it embeds as a sub-Hopf algebra of $H_L = H \otimes_R L$. We will define the Hopf dual to be a sub-Hopf algebra of $(H_L)^\circ$. Let $\mathcal{J}$ denote the set of ideal I in H such that H/I is a finitely generated R-module. Moreover, denote by $\mathcal{J}'$ the set of ideals I in H such that H/I is free of finite rank.

Definition 2.3.1. We define the Hopf dual of H to be

$$H^\circ := \{f \in H^* : f|_I = 0 \text{ for some } I \in \mathcal{J}\}.$$

By the above H° is torsion-free.

If $n \geq 0$ and $x \in H$ we have for any $f \in H^*$ that $f(x) = 0$ if and only if $f(\pi^n x) = 0$. Thus if $0 \neq f \in H^\circ$ then $f|_I = 0$ for some $I \in \mathcal{J}$ where H/I is not torsion. Moreover we then have $f|_{I_L \cap H} = 0$ and so by replacing I with $I_L \cap H$ we may in addition assume that H/I is torsion-free. Since R is a PID this shows that

$$H^\circ = \{f \in H^* : f|_I = 0 \text{ for some } I \in \mathcal{J}'\}.$$

Moreover by extending scalars we may identify H° with an R-submodule of H_L°.

Lemma 2.3.2. *H° is the set of matrix coefficients of H-modules which are free of finite rank over R. Hence it is an R-submodule of H^*.*

Proof. From the above discussion, the first part follows by the same proof as in Lemma 2.2.4. The second part follows from Lemma 2.2.2(i). □

Since the H-modules which are finite free over R are closed under taking tensor products and duals, and we can take dual bases, we have proved as in Lemma 2.2.6:

Lemma 2.3.3. *The restrictions of the Hopf algebra structure from H_L° to H° makes H° into an R-Hopf algebra. In particular the algebra maps on H° are just the dual maps of the coalgebra maps on H and vice-versa.*

Remark 2.3.4. Some of the above arguments were implicit in Lusztig's work, see [59, 7.1].

We now wish to establish some correspondence between comodules over H° and certain H-modules. We call an H-module M *locally finite* if for all $m \in M$, $H \cdot m$ is finitely generated over R.

Proposition 2.3.5. *Every H°-comodule has a canonical structure of a locally finite H-module with respect to which every comodule homomorphism is an H-modules homomorphism. In other words there is a canonical faithful embedding of categories between the category of H°-comodules and the category of locally finite H-modules.*

Proof. If M is an H°-comodule with coaction $\rho : M \to M \otimes_R H^\circ$, write $\rho(m) = \sum m_1 \otimes m_2$. Then we set

$$u \cdot m = \sum m_2(u) m_1$$

for all $u \in H$. It follows from the comodule axioms that this gives a well defined module structure, i.e. that $1 \cdot m = m$ and that $u \cdot (u' \cdot m) = (uu') \cdot m$ for all $u, u' \in H$ and all $m \in M$. Indeed, by definition of the comultiplication on H°, we have

$$(uu') \cdot m = \sum m_2(uu') m_1 = \sum m_3(u') m_2(u) m_1 = \sum m_2(u')(u \cdot m_1) = u \cdot (u' \cdot m)$$

as required. Also, since $\varepsilon \in H^\circ$ is the map $f \mapsto f(1)$, we have

$$1 \cdot m = \sum m_2(1) m_1 = \sum \varepsilon(m_2) m_1 = m$$

as required. Moreover by definition of the module structure, $H \cdot m$ is finitely generated over R for all $m \in M$.

Finally it follows from the definition of the action that any comodule homomorphism is also a module homomorphism. Indeed, if $f : M \to N$ is a comodule homomorhpism, then for all $m \in M$, we have $\rho_N(f(m)) = \sum f(m_1) \otimes m_2$. Thus we see that for all $u \in H$, we have

$$u \cdot f(m) = \sum m_2(u) f(m_1) = f\left(\sum m_2(u) m_1\right) = f(u \cdot m)$$

as required. □

Next, we want to show that the functor we just defined is full, i.e. that every H-module map between two H°-comodules is a comodule homomorphism. We first need a technical result. Suppose M is a locally finite H-module. Note that we have an R-module injection

$\phi_M : M \to \mathrm{Hom}_R(H, M)$ given by $\phi_M(m)(u) = u \cdot m$ for all $u \in H$ and $m \in M$. Moreover we have a map

$$\theta_M : M \otimes_R H^\circ \to \mathrm{Hom}_R(H, M)$$

given by $\theta_M(m \otimes f)(u) = f(u)m$. When the H-module structure on M arises from an H°-comodule structure then we have $\phi_M = \theta_M \circ \rho$. Therefore we can use this expression for ϕ_M as an alternative definition of the module structure on M. We claim that the map θ_M is injective. More generally we have the following:

Lemma 2.3.6. *Let A and B be R-modules and suppose C is an R-submodule of A^* such that A^*/C has no π-torsion. Let M be any R-module and let*

$$\theta_{M,C} : \mathrm{Hom}_R(B, M) \otimes_R C \to \mathrm{Hom}_R(A \otimes_R B, M)$$

be defined by $\theta_{M,C}(g \otimes f)(x \otimes y) = f(x)g(y)$. Then the map $\theta_{M,C}$ is injective.

Proof. Suppose that $0 \neq u = \sum_{i=1}^{s} g_i \otimes f_i \in \mathrm{Hom}_R(B, M) \otimes_R C$. The R-submodule N of $\mathrm{Hom}_R(B, M)$ generated by the g_i is finitely generated, so since R is a PID we can pick a generating set $n_1, \ldots, n_l, t_1, \ldots, t_m$ for N such that $n_1, \ldots, n_l$ are torsion-free while $t_1, \ldots, t_m$ are π-torsion, and

$$N = \bigoplus_{i=1}^{l} Rn_i \oplus \bigoplus_{j=1}^{m} Rt_j.$$

For each $1 \leq j \leq m$, let a_j be the positive integer such that $Rt_j \cong R/\pi^{a_j}R$.

Now, to show that $\theta_{M,C}(u) \neq 0$ it suffices to show that the restriction of $\theta_{M,C}$ to the span of the $n_i \otimes f_j$ and $t_k \otimes f_j$ is injective. So suppose we are given

$$v = \sum r_{ij} n_i \otimes f_j + \sum r'_{kj} t_k \otimes f_j \in \ker \theta_{M,C}.$$

Evaluating at $x \otimes y$ we get $\sum_{i,j} r_{ij} f_j(x) n_i(y) + \sum_{k,j} r'_{kj} f_j(x) t_k(y) = 0$ for all $x \in A$ and $y \in B$. In particular we have $\sum_{i,j} r_{ij} f_j(x) n_i + \sum_{k,j} r'_{kj} f_j(x) t_k = 0$ for any fixed $x \in A$. Since we have a direct sum decomposition of N it follows that

$$\sum_j r_{ij} f_j(x) = 0 \quad \text{and} \quad \sum_j r'_{kj} f_j(x) \in \pi^{a_k} R$$

for all $x \in A$ and all $1 \leq i \leq l$ and $1 \leq k \leq m$. In particular, for all k, $\sum_j r'_{kj} f_j = \pi^{a_k} g_k$ for some $g_k \in C$ since A^*/C has no π-torsion.

Therefore we have

$$\sum_j r_{ij} f_j = 0 \quad \text{and} \quad \sum_j r'_{kj} f_j \in \pi^{a_k} C,$$

and hence

$$n_i \otimes \sum_j r_{ij} f_j = 0 = t_k \otimes \sum_j r'_{kj} f_j$$

for all i, k, and so $v = 0$ as required. $\square$

Corollary 2.3.7. *Let M be an R-module.*

(i) The map $\theta_M : M \otimes_R H^\circ \to Hom_R(H, M)$ is injective.

(ii) The map $M \otimes_R H^\circ \otimes_R H^\circ \to Hom_R(H \otimes_R H, M)$ sending $m \otimes g \otimes f$ to $x \otimes y \mapsto g(x)f(y)m$ is injective.

Proof. Let $A = H$ and $C = H^\circ$. From the definition of H° it follows that A^*/C is torsion-free. Then (i) follows immediately from Lemma 2.3.6 by putting $B = R$. For (ii) note that this map is simply the composite

$$M \otimes_R H^\circ \otimes_R H^\circ \xrightarrow{\theta_M \otimes 1} \mathrm{Hom}_R(H, M) \otimes_R H^\circ \xrightarrow{\varpi} \mathrm{Hom}_R(H \otimes_R H, M)$$

where $\varpi(h \otimes f)(x \otimes y) = f(y)h(x)$. The map $\theta_M \otimes 1$ is injective by (i) and because H° is flat while the map ϖ is injective by putting $B = H$ in Lemma 2.3.6. □

We can now deduce the result we were aiming for.

Proposition 2.3.8. *The functor associating any H°-comodule to the corresponding H-module is a fully faithful embedding.*

Proof. From what we have done already we just need to show that any H-module map $f : M \to N$ between two H°-comodules is a comodule homomorphism. Write ρ_M and ρ_N for the coactions on M and N respectively, and pick $m \in M$ and $u \in H$. Then we know that $u \cdot m = \sum m_2(u)m_1$ and we have $u \cdot f(m) = \sum m_2(u)f(m_1)$ since f is a module homomorphism. On the other hand by definition of the action on N we have $u \cdot f(m) = \sum f(m)_2(u)f(m)_1$. To show that f is a comodule map we need to show that

$$\sum f(m_1) \otimes m_2 = \sum f(m)_1 \otimes f(m)_2$$

or in other words that $\rho_N \circ f = (f \otimes 1) \circ \rho_M$.

Write $\tilde{\rho}_1 = \rho_N \circ f$ and $\tilde{\rho}_2 = (f \otimes 1) \circ \rho_M$. Moreover recall the map $\phi_M : M \to \mathrm{Hom}_R(H, M)$ given by $\phi(m)(u) = u \cdot m$. Then let

$$\tilde{\phi}_M = \phi_N \circ f : M \to \mathrm{Hom}_R(H, N)$$

so that $\tilde{\phi}_M(m)(u) = u \cdot f(m)$. Then by definition $\tilde{\phi}_M = \theta_N \circ \tilde{\rho}_1$. On the other hand by our above observation we see that $\tilde{\phi}_M = \theta_N \circ \tilde{\rho}_2$. Since θ_N is injective by Corollary 2.3.7(i) the result follows. □

From now on, if M is a locally finite H-module we will say that it is an H°-comodule to mean that its H-module structure arises from an H°-comodule structure.

In order for the above functor to be an isomorphism of categories we therefore just need to show that it is surjective. This may not be true in general, however we can write a very simple necessary and sufficient condition for an isomorphism of categories to hold. Suppose M is a locally finite H-module and recall the maps $\phi_M : M \to \mathrm{Hom}_R(H, M)$ and $\theta_M : M \otimes_R H^\circ$ from before.

Proposition 2.3.9. *A locally finite H-module M is an H°-comodule if and only if $\phi_M(m)$ belongs to the image of θ_M for all $m \in M$.*

Proof. If M is a comodule with coaction ρ, then by our observation preceding Lemma 2.3.6 we have $\phi_M = \theta_M \circ \rho$ where ϕ_M comes from the induced H-module structure, and the result is clear. Conversely assume $\phi_M(m)$ belongs to the image of θ_M for all $m \in M$. Fix $m \in M$. Then there exists $m_1, \ldots, m_n \in M$ and $f_1, \ldots, f_n \in H^\circ$ such that for all $x \in H$, $x \cdot m = \sum_{i=1}^n f_i(x) m_i$ and we define

$$\rho(m) = \sum_{i=1}^n m_i \otimes f_i,$$

i.e. $\rho(m)$ is the unique element of $M \otimes_R H^\circ$ such that $\theta_M(\rho(m)) = \phi_M(m)$. We now have to check that this satisfies the comodule axioms. By definition, the counit on H° is defined by $\varepsilon(f) = f(1)$ and so

$$\big((1 \otimes \varepsilon) \circ \rho\big)(m) = \sum_{i=1}^n f_i(1) m_i = 1 \cdot m = m$$

as required. Finally we aim to show that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"] & M \otimes_R H^\circ \arrow[d, "1\otimes\Delta"] \\ M \otimes_R H^\circ \arrow[r, "\rho\otimes 1"] & M \otimes_R H^\circ \otimes_R H^\circ \end{tikzcd}$$

By Corollary 2.3.7(ii), the natural map $M \otimes_R H^\circ \otimes_R H^\circ \to \mathrm{Hom}_R(H \otimes_R H, M)$ is injective. Hence it suffices to show that $\big((1 \otimes \Delta) \circ \rho\big)(m)$ and $\big((\rho \otimes 1) \circ \rho\big)(m)$ act in the same way on $H \otimes_R H$ for all $m \in M$. But the former sends $x \otimes y$ to $(xy) \cdot m$ while the latter sends $x \otimes y$ to $x \cdot (y \cdot m)$ for any $x, y \in H$, which are clearly equal. $\square$

Since Lemma 2.3.6 was quite general, the same argument as in the above proof shows the following:

Lemma 2.3.10. *Suppose M is a locally finite H-module and let C be a subcoalgebra of H° such that H^*/C is torsion-free. If $\phi_M(m)$ belongs to the image of $\theta_{M,C}$ (where we set $B = R$ and $A = H$) for all $m \in M$ then M is a C-comodule.*

2.4 Quantized enveloping algebras and their representations

We now begin collecting basic facts about quantum groups, starting with quantized enveloping algebras. For a fuller treatment, see eg [25, 45, 27, 60].

Convention. From now on, k will denote the residue field $R/\pi R$ of R.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ contained in a Borel subalgebra. We choose a positive root system and we denote the simple roots by $\alpha_1, \ldots, \alpha_n$. Let $C = (a_{ij})$ denote the Cartan matrix. We let G be the simply connected semisimple algebraic group corresponding to $\mathfrak{g}$, and we let B be the Borel subgroup corresponding to the positive root system, and let $N \subset B$ be its unipotent radical. Let $\mathfrak{b} = \mathrm{Lie}(B)$ and $\mathfrak{n} = \mathrm{Lie}(N)$. Let W be the Weyl group of $\mathfrak{g}$, and let $\langle\,,\rangle$ denote the standard normalised W-invariant bilinear form on $\mathfrak{h}^*$. Let $P \subset \mathfrak{h}^*$ be the weight lattice and $Q \subset P$ be the root lattice. Let T_P denote the abelian group $\mathrm{Hom}_{\mathbb{Z}}(P, L^\times)$. We will use the additive notation for this group. Let d be the smallest natural number such that $\langle \mu, P\rangle \subset \frac{1}{d}\mathbb{Z}$ for all $\mu \in P$. Let $d_i = \frac{\langle \alpha_i, \alpha_i\rangle}{2} \in \{1,2,3\}$ and write $q_i := q^{d_i}$.

We make the following two assumptions for the rest of this thesis. First, we assume that $q^{\frac{1}{d}}$ exists in R and that $q^{\frac{1}{d}} \equiv 1 \pmod{\pi}$. Secondly, we assume that $p > 2$ and, if $\mathfrak{g}$ has a component of type G_2, we furthermore restrict to $p > 3$.

Notation. We need to introduce quite a bit of notation. All the above algebraic groups and Lie algebras have k-forms, and we write $G_k, \mathfrak{g}_k, \ldots$ etc to denote them.

For each $\lambda \in P$ we have an associated element of T_P sending a given $\mu \in P$ to $q^{\langle \lambda, \mu\rangle}$. We will also denote this element of T_P by λ.

For $n \in \mathbb{Z}$ and $t \in L$, we write $[n]_t := \frac{t^n - t^{-n}}{t - t^{-1}}$. We then set the quantum factorial numbers to be given by $[0]_t! = 1$ and $[n]_t! := [n]_t[n-1]_t \cdots [1]_t$ for $n \geq 1$. Then we set

$$\begin{bmatrix} n \\ i \end{bmatrix}_t := \frac{[n]_t!}{[i]_t![n-i]_t!}$$

when $n \geq i \geq 1$.

Definition 2.4.1. The simply connected quantized enveloping algebra $U_q(\mathfrak{g})$ is defined to be the L-algebra with generators $E_{\alpha_1}, \ldots, E_{\alpha_n}$, $F_{\alpha_1}, \ldots, F_{\alpha_n}$, K_λ, $\lambda \in P$, satisfying the following relations:

$$\begin{aligned}
&K_\lambda K_\mu = K_{\lambda+\mu}, \quad K_0 = 1,\\
&K_\lambda E_{\alpha_i} K_{-\lambda} = q^{\langle \lambda, \alpha_i\rangle} E_{\alpha_i}, \quad K_\lambda F_{\alpha_i} K_{-\lambda} = q^{-\langle \lambda, \alpha_i\rangle} F_{\alpha_i},\\
&[E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q_i - q_i^{-1}},
\end{aligned}$$

and the *quantum Serre relations*

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} E_{\alpha_i}^{1-a_{ij}-l} E_{\alpha_j} E_{\alpha_i}^l = 0 \quad (i \neq j), \tag{2.1}$$

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} F_{\alpha_i}^{1-a_{ij}-l} F_{\alpha_j} F_{\alpha_i}^l = 0 \quad (i \neq j). \tag{2.2}$$

We will abbreviate $U_q(\mathfrak{g})$ to U_q when no confusion can arise as to the choice of Lie algebra $\mathfrak{g}$. We can define Borel and nilpotent subalgebras, namely $U_q^{\geq 0}$ is the subalgebra generated by all the $K's$ and the $E's$, and U_q^+ is the subalgebra generated by all the $E's$. Similarly, U_q^- is defined to be the subalgebra generated by all the $F's$. There is also a Cartan subalgebra given by $U_q^0 := L[K_\lambda : \lambda \in P]$.

Proposition 2.4.2. *(i)* ([45, Lemma 4.6]) *There is a unique algebra automorphism ω of U_q defined by $\omega(E_{\alpha_i}) = F_{\alpha_i}$, $\omega(F_{\alpha_i}) = E_{\alpha_i}$ and $\omega(K_\lambda) = K_{-\lambda}$.*

(ii) ([45, Proposition 4.11]) *U_q is an L-Hopf algebra with operations given by*

$$\begin{aligned}
&\Delta(K_\lambda) = K_\lambda \otimes K_\lambda && \varepsilon(K_\lambda) = 1 && S(K_\lambda) = K_{-\lambda}\\
&\Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes 1 + K_{\alpha_i} \otimes E_{\alpha_i} && \varepsilon(E_{\alpha_i}) = 0 && S(E_{\alpha_i}) = -K_{-\alpha_i} E_{\alpha_i}\\
&\Delta(F_{\alpha_i}) = F_{\alpha_i} \otimes K_{-\alpha_i} + 1 \otimes F_{\alpha_i} && \varepsilon(F_{\alpha_i}) = 0 && S(F_{\alpha_i}) = -F_{\alpha_i} K_{\alpha_i}
\end{aligned}$$

for $i = 1, \dots, n$ and all $\lambda \in P$. Then $U_q^{\geq 0}$ is a sub-Hopf algebra of U_q.

Remark 2.4.3. We remark here that with this Hopf algebra structure on U_q, the quantum Serre relations are equivalent to:

$$\mathrm{ad}(E_{\alpha_i}^{1-a_{ij}})(E_{\alpha_j}) = \mathrm{ad}(F_{\alpha_i}^{1-a_{ij}})(F_{\alpha_j}) = 0 \quad (i \neq j).$$

This is therefore completely analogous to the Serre relations in $U(\mathfrak{g})$.

We now recall the construction of some PBW-type bases for U_q.

Theorem 2.4.4 ([45, Theorem 4.21])**.** *(i) There is a triangular decomposition*

$$U_q^- \otimes_L U_q^0 \otimes_L U_q^+ \cong U_q$$

given by the multiplication map.

(ii) The K_μ with $\mu \in P$ form a basis of U_q^0, so that U_q^0 is isomorphic to the group algebra LP.

(iii) The algebra U_q^+ (resp. U_q^-) is the algebra with generators E_{α_i} (resp. F_{α_i}), $1 \leq i \leq n$, and relations the quantum Serre relations (2.1) (resp. (2.2)).

Hence we see that in order to get a basis of U_q, it is sufficient to find bases for $U_q^{\pm}$. Moreover, since $U_q^- = \omega(U_q^+)$, it suffices to find a basis for U_q^+. To obtain that, we consider the action of the braid group on U_q due to Lusztig.

Notation. For any integer $s \geq 0$, we write $E_{\alpha_i}^{(s)} := \frac{E_{\alpha_i}^s}{[s]_{q_i}!}$ and $F_{\alpha_i}^{(s)} := \frac{F_{\alpha_i}^s}{[s]_{q_i}!}$.

Theorem 2.4.5 ([60, 39.4])**.** *For each $1 \leq i \leq n$, there is an algebra automorphism on U_q defined by*

$$\begin{aligned}
T_i E_{\alpha_i} &= -F_{\alpha_i} K_{\alpha_i}\\
T_i F_{\alpha_i} &= -K_{-\alpha_i} E_{\alpha_i}\\
T_i E_{\alpha_j} &= \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{-s} E_i^{(-a_{ij}-s)} E_j E_i^{(s)} \quad (i \neq j)\\
T_i F_{\alpha_j} &= \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{s} F_i^{(s)} F_j F_i^{(-a_{ij}-s)} \quad (i \neq j)\\
T_i K_\lambda &= K_{s_i(\lambda)}.
\end{aligned}$$

Moreover these algebra automorphisms satisfy the braid relations, meaning that

$$\underbrace{T_i T_j \cdots}_{m} = \underbrace{T_j T_i \cdots}_{m}$$

for any $i \neq j$ such that $s_{\alpha_i} s_{\alpha_j}$ has order m in W.

The above can be extended to construct automorphisms T_w for any element $w \in W$. Indeed, if $w = s_{i_1} \cdots s_{i_s}$ is a reduced expression for w, then let $T_w = T_{i_1} T_{i_2} \cdots T_{i_s}$. Moreover we write $\ell(w)$ to denote the length of any reduced expression for w.

Lemma 2.4.6 ([45, 8.18]). *The operators T_w are independent of the choice of reduced expression for w. Moreover, if $w = w_1 w_2$ where $\ell(w) = \ell(w_1) + \ell(w_2)$ then $T_w = T_{w_1} T_{w_2}$.*

Let N denote the number of positive roots of $\mathfrak{g}$. Let $w_0 \in W$ be the unique element of longest length and choose a reduced expression $w_0 = s_{i_1} \cdots s_{i_N}$. Recall that then

$$\beta_1 := \alpha_{i_1}, \beta_2 := s_{i_1}(\alpha_{i_2}), \ldots, \beta_N := s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})$$

are all the positive roots of $\mathfrak{g}$ in some order (see [43, 5.6, Exercise 1]). Then we define elements $E_{\beta_1}, \ldots, E_{\beta_N}$ of U_q by

$$E_{\beta_j} := T_{i_1} \cdots T_{i_{j-1}}(E_{\alpha_{i_j}}).$$

What we need is summarized in the following:

Theorem 2.4.7 ([45, Proposition 8.20 & Theorem 8.24]). *For each $1 \leq j \leq N$, $E_{\beta_j} \in U_q^+$ and if furthermore $\beta_j = \alpha$ is a simple root, then we have $E_{\beta_j} = E_\alpha$. Finally, the set of all ordered monomials*

$$E_{\beta_1}^{m_1} \cdots E_{\beta_N}^{m_N}$$

forms a basis for U_q^+.

This basis for U_q^+ depends on the choice of reduced expression for w_0, so we fix one once and for all from this point. Also note that we have in general $K_\lambda E_{\beta_j} K_{-\lambda} = q^{\langle \lambda, \beta_j \rangle} E_{\beta_j}$ for every $\lambda \in P$ and $1 \leq j \leq N$, see [45, 8.18(4)].

We now let $F_{\beta_j} := \omega(E_{\beta_j})$. By the previous Theorem, the corresponding monomials

$$F_{\beta_1}^{m_1} \cdots F_{\beta_N}^{m_N}$$

form a basis of U_q^-. The triangular decomposition then immediately gives a basis for U_q, namely:

Corollary 2.4.8 (PBW Theorem for U_q). *The set of all ordered monomials*

$$F_{\beta_1}^{n_1} \cdots F_{\beta_N}^{n_N} K_\lambda E_{\beta_1}^{m_1} \cdots E_{\beta_N}^{m_N}$$

for $m_i, n_j \in \mathbb{Z}_{\geq 0}$ and $\lambda \in P$, is an L-basis for U_q.

For short we will write

$$M_{\boldsymbol{r},\boldsymbol{s},\lambda} := \boldsymbol{F}^{\boldsymbol{r}} K_\lambda \boldsymbol{E}^{\boldsymbol{s}}$$

where $\boldsymbol{r}, \boldsymbol{s} \in \mathbb{Z}_{\geq 0}^N$. The *height* of such a monomial is defined to be

$$\operatorname{ht}(M_{\boldsymbol{r},\boldsymbol{s},\lambda}) := \sum_{j=1}^{N} (r_j + s_j) \operatorname{ht}(\beta_j)$$

where $\operatorname{ht}(\beta) := \sum_{i=1}^n a_i$ for a positive root $\beta = \sum_i a_i \alpha_i$.

Lemma 2.4.9 ([25, I.6.11])**.** *The height gives rise to a positive algebra filtration on U_q defined by*

$$F_i U_q := L\text{-}span\{M_{\boldsymbol{r},\boldsymbol{s},\lambda} : \operatorname{ht}(M_{\boldsymbol{r},\boldsymbol{s},\lambda}) \leq i\}.$$

From now on we will always refer to this filtration as the *height filtration* on U_q. It can be extended to a multifiltration as follows (see [31, Section 10.1] and [25, I.6.11]): the associated graded algebra $U^{(1)} = \operatorname{gr} U_q$ with respect to the above filtration has the same presentation as U_q, with the exception that now all the E's commute with all the F's. Moreover it has the same vector space basis, by which we mean the basis for $U^{(1)}$ is consists of the symbols of the basis elements for U_q. If we impose the reverse lexicographic ordering on $\mathbb{Z}_{\geq 0}^{2N}$, then we can filter $U^{(1)}$ by assigning to each monomial $M_{\boldsymbol{r},\boldsymbol{s},\lambda}$ the degree $(r_1, \ldots, r_N, s_1, \ldots, s_N)$. In other words, for each $\boldsymbol{d} \in \mathbb{Z}_{\geq 0}^{2N}$, we set $F_{\boldsymbol{d}} U^{(1)}$ to be the span of the monomials $M_{\boldsymbol{r},\boldsymbol{s},\lambda}$ such that $(r_1, \ldots, r_N, s_1, \ldots, s_N) \leq \boldsymbol{d}$. This is an algebra multifiltration, and we denote the corresponding associated graded algebra of $U^{(1)}$ by $U^{(2N+1)}$.

Theorem 2.4.10 ([31, Proposition 10.1])**.** *The algebra $U^{(2N+1)}$ is the L-algebra with generators*

$$E_{\beta_1}, \ldots, E_{\beta_N}, F_{\beta_1}, \ldots, F_{\beta_N}, K_\lambda (\lambda \in P)$$

and relations

$$\begin{aligned}
&K_\lambda K_\mu = K_{\lambda+\mu}, \quad K_0 = 1, \\
&K_\lambda E_{\beta_i} = q^{\langle \lambda, \beta_i \rangle} E_{\beta_i} K_\lambda, \quad K_\lambda F_{\beta_j} = q^{-\langle \lambda, \beta_j \rangle} F_{\beta_j} K_\lambda, \\
&E_{\beta_i} F_{\beta_j} = F_{\beta_j} E_{\beta_i} \\
&E_{\beta_i} E_{\beta_j} = q^{\langle \beta_i, \beta_j \rangle} E_{\beta_j} E_{\beta_i}, \quad F_{\beta_i} F_{\beta_j} = q^{\langle \beta_i, \beta_j \rangle} F_{\beta_j} F_{\beta_i}
\end{aligned}$$

for $\lambda, \mu \in P$ and $1 \leq i, j \leq N$. In particular, $U^{(2N+1)}$ and U_q are Noetherian domains.

We now recall details about Lusztig's integral form of U_q. It is defined to be the R-subalgebra U^{res} of U_q generated by K_λ ($\lambda \in P$) and all $E_{\alpha_i}^{(r)}$ and $F_{\alpha_i}^{(r)}$ for $r \geq 0$ and $1 \leq i \leq n$.

Notation. For $1 \leq i \leq n$, $c, t \in \mathbb{Z}$ with $t \geq 0$ we define

$$\begin{bmatrix} K_{\alpha_i}; c \\ t \end{bmatrix} = \prod_{j=1}^{t} \frac{K_{\alpha_i} q_i^{c-j+1} - K_{\alpha_i}^{-1} q_i^{-(c-j+1)}}{q_i^j - q_i^{-j}}.$$

Remark 2.4.11. Before we go on any further, we remark here that many results to do with integral forms in the literature are not usually set in our p-adic context. Instead the integral forms are typically defined over the Laurent polynomial ring $\mathbb{Z}[v, v^{-1}]$ or the localised ring $\mathbb{Z}[v]_{\mathfrak{m}}$ where $\mathfrak{m}$ is the ideal generated by $v-1$ and a fixed prime p. That latter choice is for instance the setting used by Andersen, Polo and Wen [4], which we will use extensively. In general, the base ring does not play a big role in the structure of these integral forms, and the general results of Lusztig [59] hold identically. Similarly, the work of Andersen, Polo and Wen only uses the property of $\mathbb{Z}[v]_{\mathfrak{m}}$ that it is a commutative Noetherian local ring. Since R shares the same properties (it is even nicer since it's a PID), all the statements in [4] also hold over R with identical proofs.

By [45, 11.1, p.238] we have that all such $\left[\begin{smallmatrix} K_{\alpha_i};c \\ t \end{smallmatrix}\right]$ lie in U^{res}. We set $(U^{\text{res}})^0$ to be the R-subalgebra of U^{res} generated by all K_λ and all $\left[\begin{smallmatrix} K_{\alpha_i};c \\ t \end{smallmatrix}\right]$. Similarly we set $(U^{\text{res}})^+$ (resp. $(U^{\text{res}})^-$) to be the R-subalgebra generated by all $E_{\alpha_i}^{(r)}$ (resp. $F_{\alpha_i}^{(r)}$) for $r \geq 0$ and $1 \leq i \leq n$. Finally we let $U^{\text{res}}(\mathfrak{b})$ denote the R-subalgebra generated by $(U^{\text{res}})^0$ and all $E_{\alpha_i}^{(r)}$ for $r \geq 0$ and $1 \leq i \leq n$.

Lusztig has shown that his integral form has a triangular decomposition and a PBW type basis analogous to the one for U_q (see [59, Theorem 6.7]). However he was working with the *adjoint form* of the quantum group, i.e. it's the L-subalgebra of U_q with the same generators and relations except that you only take K_λ with $\lambda \in Q$ as generators. We briefly sketch how to extend this to U^{res}. Write $U_{\text{ad}}^{\text{res}}$ for the adjoint form of Lusztig's integral form, i.e. the R-subalgebra of U_q generated by K_λ ($\lambda \in Q$) and all $E_{\alpha_i}^{(r)}$ and $F_{\alpha_i}^{(r)}$ for $r \geq 0$ and $1 \leq i \leq n$. Note that $(U_{\text{ad}}^{\text{res}})^{\pm} = (U^{\text{res}})^{\pm}$.

First, the triangular decomposition $U^{\text{res}} \cong (U^{\text{res}})^- \otimes_R (U^{\text{res}})^0 \otimes_R (U^{\text{res}})^+$ follows from the one for U_q: from the defining relations of U_q, there is clearly a surjection $(U^{\text{res}})^- \otimes_R (U^{\text{res}})^0 \otimes_R (U^{\text{res}})^+ \to U^{\text{res}}$ given by the multiplication map, but this map is also injective since it is injective after extending scalars to L. Next, we have that the sets of monomials

$$E_{\beta_1}^{(m_1)} \cdots E_{\beta_N}^{(m_N)} \quad \text{and} \quad F_{\beta_1}^{(m_1)} \cdots F_{\beta_N}^{(m_N)}$$

form bases of $(U_{\text{ad}}^{\text{res}})^{\pm} = (U^{\text{res}})^{\pm}$ by [59, Theorem 6.7]. Finally, Lusztig also showed in *loc. cit.* that $(U_{\text{ad}}^{\text{res}})^0$ has basis given by the products

$$u_{\boldsymbol{\delta},\boldsymbol{t}} := \prod_{i=1}^{n} K_{\alpha_i}^{\delta_i} \begin{bmatrix} K_{\alpha_i}; 0 \\ t_i \end{bmatrix}$$

where $t_i \geq 0$ and $\delta_i \in \{0,1\}$ for all $1 \leq i \leq n$. Now, let $\lambda_1 = 0, \ldots, \lambda_{m'}$ be a set of coset representatives for P/Q. We claim that the elements $K_{\lambda_i} u_{\boldsymbol{\delta},\boldsymbol{t}}$ form an R-basis for $(U^{\text{res}})^0$. Indeed, there is an isomorphism of R-modules

$$R(P/Q) \otimes_R (U_{\text{ad}}^{\text{res}})^0 \to (U^{\text{res}})^0$$

given by $\lambda_i \otimes u \mapsto K_{\lambda_i} \cdot u$. This is easily seen to be surjective: for every $\mu \in P$, there is a unique λ_i such that $\mu - \lambda_i \in Q$, and so $(U^{\text{res}})^0$ is generated by the K_{λ_i} as a module over $(U_{\text{ad}}^{\text{res}})^0$. Now the map is injective since the left hand side is π-torsion free and after extending scalars to L, the map becomes the canonical isomorphism of vector spaces $LP \cong L(P/Q) \otimes_L LQ$. Thus we have shown:

Theorem 2.4.12. *The R-algebra U^{res} is free over R.*

We now turn to representations. Suppose that V is a one dimensional U_q-module. From the relations $K_\lambda E_{\alpha_i} K_{-\lambda} = q^{\langle \lambda, \alpha_i \rangle} E_{\alpha_i}$ and $K_\lambda F_{\alpha_i} K_{-\lambda} = q^{-\langle \lambda, \alpha_i \rangle} F_{\alpha_i}$, we see that E_{α_i} and F_{α_i} act as zero for every $1 \leq 1 \leq n$ because q is not a root of unity. Moreover, by the commutator relation $[E_{\alpha_i}, F_{\alpha_i}] = \frac{K_{\alpha_i} - K_{-\alpha_i}}{q_i - q_i^{-1}}$, we see that $K_{\alpha_i}^2 = K_{2\alpha_i}$ acts as the identity. Hence the one dimensional representations of U_q are parametrized by the group Σ of characters of the quotient $P/2Q$. Let Σ_R denote the group of R-valued characters of $P/2Q$.

Remark 2.4.13. We quickly remark here that for as long as p does not divide $m := |P/2Q|$ and if k contains all m-th roots of unity, it then follows from Hensel's lemma that R contains all m-th roots of unity and hence we have that $\Sigma = \Sigma_R$.

By [4, Lemma 1.1], for each $\lambda \in P$ and $\sigma \in \Sigma_R$, there is a unique character $\psi_{\lambda,\sigma}$: $(U^{\text{res}})^0 \to R$ defined by

$$\psi_{\lambda,\sigma}(K_\mu) = \sigma(\mu) q^{\langle \lambda, \mu \rangle} \quad \text{and} \quad \psi_{\lambda,\sigma}\left(\begin{bmatrix} K_{\alpha_i}; c \\ t \end{bmatrix} \right) = \sigma(\alpha_i)^t \begin{bmatrix} \langle \lambda, \alpha_i^\vee \rangle + c \\ t \end{bmatrix}_{q_i} \tag{2.3}$$

for all $\mu \in P$, $1 \leq i \leq n$, $c \in \mathbb{Z}$ and $t \in \mathbb{Z}_{\geq 0}$. This is just a twist by σ of the element $\lambda \in T_P$ that we described before. We will say that the characters $\psi_{\lambda,\sigma}$ for varying λ and fixed σ are of type σ. When σ is trivial, we write $\psi_{\lambda,\sigma}$ simply as ψ_λ and say that it is of type **1**.

Analogously, for $\lambda \in P$ and $\sigma \in \Sigma$, we may define a character $\psi_{\lambda,\sigma}$ of U_q^0 by setting

$$\psi_{\lambda,\sigma}(K_\mu) = \sigma(\mu) q^{\langle \lambda, \mu \rangle}$$

for all $\mu \in P$.

Given a $(U^{\text{res}})^0$-module M, $\sigma \in \Sigma_R$ and $\lambda \in P$, we let

$$M_{\lambda,\sigma} = \{ m \in M : um = \psi_{\lambda,\sigma}(u) m \quad \text{for all } u \in (U^{\text{res}})^0 \}.$$

The set $M_{\lambda,\sigma}$ is called the (λ, σ)-*weight space* of M. Given a U_q^0-module V, $\sigma \in \Sigma$ and $\lambda \in P$, we may analogously define the (λ, σ)-weight space of V as the set of $v \in V$ such that $uv = \psi_{\lambda,\sigma}(u) v$ for all $u \in U_q^0$.

Definition 2.4.14 ([4, 1.6])**.** A U^{res}-module M is said to be *integrable* if it is the sum of its weight spaces as described above and if in addition, for every $m \in M$, there is $r >> 0$ such that m is killed by $E_{\alpha_i}^{(r)}$ and $F_{\alpha_i}^{(r)}$ for any $1 \leq i \leq n$. We say it is of type **1** if all of its non-zero weight spaces are associated to characters of type **1**. We also define a $U^{\text{res}}(\mathfrak{b})$-module to be integrable if it is the sum of its weight spaces and if for every $m \in M$, $E^{(r)} m = 0$ for $r >> 0$.

Given a U_q-module V, we say that V is *integrable* if it is the sum of its weight spaces and if, for every $v \in V$, there is $r >> 0$ such that v is killed by $E_{\alpha_i}^r$ and $F_{\alpha_i}^r$ for any $1 \leq i \leq n$. We say it is of type **1** if all of its weight spaces are associated to characters of type **1**. We analogously define a $U_q^{\geq 0}$-module to be integrable if it is the sum of its weight spaces and if the E_{α_i} act nilpotently.

Lemma 2.4.15 ([4, Note added in proof p.59]). *These categories of integrable modules over these various quantum algebras are all abelian, and in fact integrable modules are closed under taking submodules.*

For each $\lambda \in T_P$, there is a Verma module M_λ which is the cyclic U_q-module with a single generator v and relations

$$E_{\alpha_i} v = 0, \quad K_\mu v = \lambda(\mu) v$$

for all $1 \le i \le n$ and $\mu \in P$. The standard representation theory of U_q is summarized below:

Theorem 2.4.16. *We continue with the above notation.*

(i) ([45, Proposition 5.1]) *Every finite dimensional U_q-module is integrable.*

(ii) ([45, 5.5 & Theorem 5.10]) *There is a bijection $\lambda \leftrightarrow V(\lambda)$ between the set P^+ of dominant integral weights and the set of finite dimensional irreducible representations of U_q of type $\mathbf{1}$. Moreover, $V(\lambda)$ is the unique irreducible quotient of M_λ.*

(iii) ([45, Theorem 5.15 & Section 6.26]) *For each $\lambda \in P^+$, $V(\lambda)$ satisfies the Weyl character formula.*

(iv) ([45, Theorem 5.17 & Section 6.26]) *Every finite dimensional U_q-module is completely reducible.*

Convention. The category of representations of U_q of type σ is equivalent to the category of representations of type $\mathbf{1}$ by [45, 5.2], hence we will only consider representations of type $\mathbf{1}$. Therefore, we will often just say 'integrable' to mean 'integrable of type $\mathbf{1}$'.

We will need to work with a second integral form U of U_q, called the De Concini-Kac integral form. This is defined to be the R-subalgebra of U_q generated by all E_{α_i}, F_{α_i}, and K_λ ($1 \le i \le n$ and $\lambda \in P$). This algebra has a similar presentation as U_q: if we write $[K_{\alpha_i}; m] := \left[\begin{smallmatrix} K_{\alpha_i}; m \\ 1 \end{smallmatrix}\right]$ for $m \in \mathbb{Z}$ and $1 \le i \le n$, then U is generated as an R-algebra by all E_{α_i}, F_{α_i}, $[K_{\alpha_i}; 0]$, and K_λ ($1 \le i \le n$ and $\lambda \in P$), with the same relations as U_q except that the commutator relation between E_{α_i} and F_{α_j} is replaced by the two relations

$$\begin{aligned} &[E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij}[K_{\alpha_i}; 0], \\ &(q_i - q_i^{-1})[K_{\alpha_i}; 0] = K_{\alpha_i} - K_{\alpha_i}^{-1}. \end{aligned}$$

Note that U is an R-Hopf algebra. Indeed we see from Proposition 2.4.2(ii) that its generators are sent to $U \otimes_R U$ by the comultiplication in U_q. Note that we have the identity

$$\Delta([K_{\alpha_i}; 0]) = [K_{\alpha_i}; 0] \otimes K_{\alpha_i} + K_{\alpha_i}^{-1} \otimes [K_{\alpha_i}; 0].$$

Note also that we have the equality

$$[K_{\alpha_i}; m] = [K_{\alpha_i}; 0] q_i^{-m} + K_{\alpha_i}[m]_{q_i}$$

for all $m \in \mathbb{Z}$, and so U contains all $[K_{\alpha_i}; m]$.

We observe that both U and U^{res} are π-adically separated since $U \subset U^{\text{res}}$ and U^{res} is free over R.

Finally, we describe the *finite part* of U_q. While classically the adjoint action of $U(\mathfrak{g})$ on itself is locally finite, this is not true for U_q. So we define the finite part of U_q to be

$$U_q^{\text{fin}} = \{u \in U_q : \dim_L \text{ad}(U_q)(u) < \infty\}.$$

U_q^{fin} is then the largest integrable submodule of U_q with respect to the adjoint action. It is a subalgebra of U_q (see [50, Corollary 2.3]). It is in fact quite large:

Lemma 2.4.17. *For every* $1 \leq i \leq n$, $K_{-2\varpi_i}$, $K_{-2\varpi_i}E_{\alpha_i}$, $K_{-2\varpi_i+\alpha_i}F_{\alpha_i} \in U_q^{fin}$.

Proof. Using the fact that q is not a root of unity, a quick computation shows that $K_{-2\varpi_i}E_{\alpha_i}$ and $K_{-2\varpi_i+\alpha_i}F_{\alpha_i}$ are non-zero scalar multiples of $\text{ad}(E_{\alpha_i})(K_{-2\varpi_i})$ and $\text{ad}(F_{\alpha_i})(K_{-2\varpi_i})$ respectively, so it's enough to show that $K_{-2\varpi_i} \in U_q^{\text{fin}}$. Now, using the fact that $\langle \varpi_i, \alpha_j \rangle = 0$ if $i \neq j$, another quick computation shows that

$$\text{ad}(E_{\alpha_j})(K_{-2\varpi_i}) = \text{ad}(F_{\alpha_j})(K_{-2\varpi_i}) = 0 \quad \forall j \neq i,$$

and

$$\text{ad}(E_{\alpha_i})^2(K_{-2\varpi_i}) = \text{ad}(F_{\alpha_i})^2(K_{-2\varpi_i}) = 0.$$

For each j, let U_j be the L-subalgebra of U_q generated by $E_{\alpha_j}, F_{\alpha_j}, K_{\alpha_j}$. Then the above shows that $\text{ad}(U_j)(K_{-2\varpi_i})$ is finite dimensional for every j by [50, Lemma 6.2]. But this is equivalent to saying that $\text{ad}(U_q)(K_{-2\varpi_i})$ is finite dimensional by [50, Proposition 6.5]. □

Remark 2.4.18. A completely analogous computation was made in [14, Lemma 2.3] working with the right adjoint action rather than the left adjoint action.

Note that the above lemma implies that the natural surjection $U_q \to M_\lambda$ restricts to a surjection $U_q^{\text{fin}} \to M_\lambda$. Now working with integral forms, the R-Hopf algebra U^{res} acts on itself via the adjoint action and, moreover, this action preserves U (see [82, Lemma 1.2]).

Definition 2.4.19. We define the *finite part* of U to be

$$U^{\text{fin}} = \{u \in U : \text{ad}(U^{\text{res}})(u) \text{ is finitely generated over } R\}.$$

Note that $U^{\text{fin}} = U_q^{\text{fin}} \cap U$. Indeed, clearly the right hand side contains U^{fin}. Conversely, if $u \in U_q^{\text{fin}} \cap U$ then $M := \text{ad}(U^{\text{res}})(u) \subset U$ is π-adically separated since U is, and $M \otimes_R L \cong \text{ad}(U_q)(u)$ which by definition is finite dimensional over L. But this forces M to be finitely generated over R (see [6, Proposition 2.7]).

2.5 Quantized coordinate rings, their comodules and the induction functor

We now recall the construction of the quantized coordinate algebra $\mathcal{O}_q$ of the simply-connected algebraic group G.

Definition 2.5.1. The quantized coordinate ring $\mathcal{O}_q$ is defined to be the subset of the Hopf dual U_q° consisting of the matrix coefficients of the finite dimensional U_q-modules of type **1**.

Lemma 2.5.2 ([25, Lemma I.7.3])**.** *$\mathcal{O}_q$ is a sub-Hopf algebra of U_q°, with Hopf algebra maps given by:*

$$\varepsilon(c_{f_i,v_j}^{V(\lambda)}) = f_i(v_j) = \delta_{ij}, \quad S(c_{f_i,v_j}^{V(\lambda)}) = c_{v_j,f_i}^{V(\lambda)^*}, \quad \Delta(c_{f_i,v_j}^{V(\lambda)}) = \sum_k c_{f_i,v_k}^{V(\lambda)} \otimes c_{f_k,v_j}^{V(\lambda)} \tag{2.4}$$

where we have $V(\lambda)^* \cong V(-w_0\lambda)$.

Proof. Since the finite dimensional U_q-modules of type **1** are closed under taking tensor products, direct sums and duals, this follows directly from Lemma 2.2.2. The isomorphism $V(\lambda)^* \cong V(-w_0\lambda)$ is [45, Proposition 5.16]. □

Proposition 2.5.3 ([25, Proposition I.7.8 & Theorem I.8.16])**.** *As an L-algebra, $\mathcal{O}_q$ is generated by the matrix coefficients of the modules $V(\varpi_1), \dots, V(\varpi_n)$. Moreover it is a domain.*

Next, we describe certain q-commutator relations in $\mathcal{O}_q$. For each i, we let B_i denote a weight basis of $V(\varpi_i)$ and B_i^* denote the dual basis. By the above $\mathcal{O}_q$ is generated by the set

$$X = \{c_{f,v}^{V(\varpi_i)} : i = 1, \dots n, f \in B_i^*, v \in B_i\}. \tag{2.5}$$

From [25, I.8.16-I.8.18], we may order X into a list $x_1, \dots, x_r$ so that there exists $q_{ij} \in R^\times$, equal to some power of q, and $\alpha_{ij}^{st}, \beta_{ij}^{st} \in L^\times$ such that

$$x_i x_j = q_{ij} x_j x_i + \sum_{s=1}^{j-1} \sum_{t=1}^{r} (\alpha_{ij}^{st} x_s x_t + \beta_{ij}^{st} x_t x_s) \tag{2.6}$$

for $1 \le j < i \le r$.

One can use these relations to deduce that $\mathcal{O}_q$ is Noetherian. Indeed let $F.$ denote the filtration on $\mathcal{O}_q$ obtained by giving x_i degree $d_i = 2^r - 2^{r-i}$. That is we set

$$F_t\mathcal{O}_q = L\text{-span}\{x_{i_1} \cdots x_{i_n} : \sum_{j=1}^{n} d_{i_j} \le t\}.$$

These degrees are chosen so that whenever $i > j > s$ and $t \le r$, we always have $d_i + d_j > d_s + d_t$. Indeed, we then have

$$\begin{aligned} d_i + d_j - d_s - d_t &= (2^{r-s} - 2^{r-j} - 2^{r-i}) + 2^{r-t} \\ &> 2^{r-s} - 2^{r-j} - 2^{r-i} \\ &> 2^{r-s} - 2^{r-j} - 2^{r-j} \\ &= 2^{r-s} - 2^{r-j+1} \ge 0 \end{aligned}$$

as required. Then we have:

Theorem 2.5.4 ([25, Proposition I.8.17 & Theorem I.8.18])**.** *With respect to the above filtration, $gr\mathcal{O}_q$ is a q-commutative L-algebra and so is Noetherian.*

Here we used the following (recall we assumed that $q^{\frac{1}{d}} \in R$):

Definition 2.5.5. Let A be an R-algebra. We say that a set of elements $x_1, \ldots, x_m \in A$ *q-commute* if for all $1 \leq i, j \leq m$ we have $x_i x_j = q^{n_{ij}} x_j x_i$ for some $n_{ij} \in \frac{1}{d}\mathbb{Z}$. Suppose that S is an R-subalgebra of A. We say that A is a *q-commutative S-algebra* if A is finitely generated over S by elements $x_1, \ldots, x_m$ which normalise S and which q-commute.

Lemma 2.5.6. *Let A be a q-commutative S-algebra as above. If S is left (resp. right) Noetherian then so is A.*

Proof. This follows immediately by induction from a noncommutative analogue of Hilbert's basis theorem (see [63, Theorem 1.2.10]). □

There is also a quantized coordinate algebra $\mathcal{O}_q(B)$ of the Borel: since $U_q^{\geq 0}$ is a Hopf-subalgebra of U_q, the restriction maps yields a Hopf algebra homomorphism $\mathcal{O}_q \to (U_q^{\geq 0})^\circ$ and we let $\mathcal{O}_q(B)$ denote its image.

We now recall how the integral forms of $\mathcal{O}_q$ and $\mathcal{O}_q(B)$ are defined. Let $\mathcal{J}$ denote the set of ideal I in U^{res} such that U^{res}/I is a finite free R-module. We now consider the set $\mathscr{I}$ consisting of ideals $I \in \mathcal{J}$ such that $I \cap (U^{\text{res}})^0$ contains a finite intersection of ideals $\ker(\psi_\lambda)$.

From [4, Definition 1.10], there is a so-called induction functor from the trivial subalgebra which is defined as follows. Let M be an R-module. Then U^{res} acts on $\text{Hom}_R(U^{\text{res}}, M)$ by $(u \cdot f)(x) = f(xu)$ for all $x, y \in U^{\text{res}}$, and we let

$$H(M) = \left\{ f \in \bigoplus_{\lambda \in P} \text{Hom}_R(U^{\text{res}}, M)_\lambda : E_{\alpha_i}^{(r)} f = F_{\alpha_i}^{(r)} f = 0,\ 1 \leq i \leq n,\ r >> 0 \right\}.$$

In other words $H(M)$ is the largest integrable submodule of $\text{Hom}_R(U^{\text{res}}, M)$.

Definition 2.5.7. The integral form of the quantized coordinate algebra is defined to be $\mathcal{A}_q := H(R)$.

By [4, Corollary 1.30], we have $f \in H(M)$ if and only if f kills an ideal $I \in \mathscr{I}$. In particular,

$$\mathcal{A}_q = \{f \in (U^{\text{res}})^* : f|_I = 0 \text{ for some } I \in \mathscr{I}\}.$$

So $\mathcal{A}_q$ is a sub-Hopf algebra of $(U^{\text{res}})^\circ$ and it may be viewed as the algebra of matrix coefficients of finite free integrable U^{res}-modules of type $\mathbf{1}$ (see [4, 1.34]). Hence we have that $\mathcal{A}_q \otimes_R L \cong \mathcal{O}_q$, because every finite dimensional U_q-module of type $\mathbf{1}$ contains an integrable U^{res}-lattice which is free of finite rank over R (see [57, Proposition 4.2]). Moreover, by [4, Theorem 1.33], $\mathcal{A}_q$ is free over R.

Next, we look at the Borel subalgebra $U^{\text{res}}(\mathfrak{b})$ of U^{res}. Let $\mathcal{I}$ be the set of $f \in \mathcal{A}_q$ such that $f|_{U^{\text{res}}(\mathfrak{b})} = 0$. The Hopf algebra homomorphism $\mathcal{A}_q \to U^{\text{res}}(\mathfrak{b})^\circ$ given by restriction has kernel precisely $\mathcal{I}$ and so we see that $\mathcal{I}$ is a Hopf ideal and that $\mathcal{B}_q := \mathcal{A}_q/\mathcal{I} \subseteq U^{\text{res}}(\mathfrak{b})^\circ$

is a Hopf algebra. Similarly to the above, [4] defined an induction functor from the trivial subalgebra to $U^{\mathrm{res}}(\mathfrak{b})$ in a completely analogous way: if M is an R-module, we define $\mathcal{H}(M)$ to be the largest integrable submodule of $\mathrm{Hom}_R(U^{\mathrm{res}}(\mathfrak{b}), M)$. By [4, Proposition 2.7(ii) and (iii)] we have that $\mathcal{B}_q = \mathcal{H}(R)$ and so it is integrable, and it is free as an R-module. Moreover we have again that $f \in \mathcal{H}(M)$ if and only if f kills an ideal I of $U^{\mathrm{res}}(\mathfrak{b})$ such that $U^{\mathrm{res}}(\mathfrak{b})/I$ is finitely generated and $I \cap (U^{\mathrm{res}})^0$ contains a finite intersection of ideals $\ker(\psi_\lambda)$.

Note that for any R-module M there are natural maps $\theta_{M,\mathcal{A}_q} : M \otimes_R \mathcal{A}_q \to \mathrm{Hom}_R(U^{\mathrm{res}}, M)$ and $\theta_{M,\mathcal{B}_q} : M \otimes_R \mathcal{B}_q \to \mathrm{Hom}_R(U^{\mathrm{res}}(\mathfrak{b}), M)$ defined in Lemma 2.3.6 (with $B = R$ and $A = U^{\mathrm{res}}$, respectively $A = U^{\mathrm{res}}(\mathfrak{b})$). The functors H and $\mathcal{H}$ both satisfy the following useful property:

Lemma 2.5.8 ([4, Theorem 1.31(iii) & Proposition 2.7(iv)]). *Let M be an R-module. Then the maps $\theta_{M,\mathcal{A}_q}$ and $\theta_{M,\mathcal{B}_q}$ are isomorphisms onto $H(M)$ and $\mathcal{H}(M)$ respectively.*

Example 2.5.9. We give an explicit description of $\mathcal{A}_q$ and $\mathcal{O}_q$ when $G = \mathrm{SL}_n$. Let $V = V(\varpi_1)$ be the natural representation of $U_q = U_q(\mathfrak{sl}_n)$. This has a heighest weight vector x_1 say, and consider $W = U^{\mathrm{res}} \cdot x_1$, the cyclic U^{res}-submodule of V generated by x_1. By [4, Section 12.5], this U^{res}-module is free of finite rank over R, with a weight basis $x_1, \ldots, x_n$ where x_i has weight $\varpi_i - \varpi_{i-1}$ (under the convention that $\varpi_0 = \varpi_n = 0$), $x_{i+1} = F_{\alpha_i} x_i$ and $E_{\alpha_i} x_{i+1} = x_i$ for all $1 \le i \le n-1$. Write X_{ij} for the matrix coefficient $C^W_{f_i,x_j} \in \mathcal{A}_q$.

Then by [4, Proposition 12.12], $\mathcal{A}_q$ is the R-algebra with generators X_{ij}, $1 \le i,j \le n$, and relations:

$$\begin{aligned} X_{il}X_{jl} &= qX_{jl}X_{il} \text{ for all } l \text{ and } i<j \\ X_{li}X_{lj} &= qX_{lj}X_{li} \text{ for all } l \text{ and } i<j \\ X_{li}X_{mj} &= X_{mj}X_{li} \text{ if } l<m \text{ and } i>j \\ X_{li}X_{mj} - X_{mj}X_{li} &= (q-q^{-1})X_{lj}X_{mi} \text{ if } l<m \text{ and } i<j \end{aligned}$$

and

$$\sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} X_{\sigma(1)1} \cdots X_{\sigma(n)n} = 1.$$

Since $\mathcal{O}_q = \mathcal{A}_q \otimes_R L$, it follows that $\mathcal{O}_q$ is the L-algebra with same generators and relations. Both $\mathcal{A}_q$ and $\mathcal{O}_q$ are then quotients of iterated skew-polynomial algebras, so that they are Noetherian, see [25, Theorems I.2.7 & I.2.10].

Remark 2.5.10. To the best of our knowledge, while the above shows that $\mathcal{A}_q$ is Noetherian in type A, it is not known in general whether $\mathcal{A}_q$ is Noetherian (unlike $\mathcal{O}_q$ which is always Noetherian by Theorem 2.5.4).

We now recall how the category of $\mathcal{A}_q$-comodules (respectively $\mathcal{B}_q$-comodules) can be identified with integrable U^{res}-modules (respectively $U^{\mathrm{res}}(\mathfrak{b})$-modules). We expect this to be well-known but we did not find a suitable reference for it, so we provide proofs.

Lemma 2.5.11. *If M is torsion-free as an R-module then $Hom_R(U^{res}, M)/H(M)$ and $Hom_R(U^{res}(\mathfrak{b}), M)/\mathcal{H}(M)$ are torsion-free. In particular $(U^{res})^*/\mathcal{A}_q$ and $U^{res}(\mathfrak{b})^*/\mathcal{B}_q$ are torsion free.*

Proof. If $\pi^n f \in \mathrm{Hom}_R(U^{\mathrm{res}}, M)$ kills an ideal in $\mathscr{I}$, then so does f as M is torsion-free. An analogous argument applies to $\mathcal{H}(M)$. The last part follows by putting $M = R$. □

Since $\mathcal{A}_q$ and $\mathcal{B}_q$ are sub-Hopf algebras of $(U^{\mathrm{res}})^\circ$ and $U^{\mathrm{res}}(\mathfrak{b})^\circ$ respectively, it follows that any comodule over $\mathcal{A}_q$ (respectively $\mathcal{B}_q$) is a comodule over $(U^{\mathrm{res}})^\circ$ (respectively $U^{\mathrm{res}}(\mathfrak{b})^\circ$). Thus we may view comodules over $\mathcal{A}_q$ and $\mathcal{B}_q$ as locally finite modules over U^{res} and $U^{\mathrm{res}}(\mathfrak{b})$ respectively. This defines functors from the categories of $\mathcal{A}_q$-comodules and $\mathcal{B}_q$-comodules to the categories of locally finite U^{res}-modules and $U^{\mathrm{res}}(\mathfrak{b})$-modules respectively.

Theorem 2.5.12. *The category of $\mathcal{A}_q$-comodules, respectively $\mathcal{B}_q$-comodules, is isomorphic to the category of integrable U^{res}-modules, respectively $U^{res}(\mathfrak{b})$-modules.*

Proof. We first show that the above functors are fully faithful. This is the exact same argument as in Proposition 2.3.8, using Lemma 2.3.6 with $A = U^{\mathrm{res}}$, $B = R$ and $C = \mathcal{A}_q$ for $\mathcal{A}_q$-comodules and with $A = U^{\mathrm{res}}(\mathfrak{b})$, $B = R$ and $C = \mathcal{B}_q$ for $\mathcal{B}_q$-comodules. For these to apply we need to show that $(U^{\mathrm{res}})^*/\mathcal{A}_q$ and $U^{\mathrm{res}}(\mathfrak{b})^*/\mathcal{B}_q$ are torsion-free, but this is just Lemma 2.5.11.

Now suppose that M is an integrable U^{res}-module. Then for all $m \in M$, the action map $\phi_M(m) : x \mapsto x \cdot m$ belongs to $H(M)$. So by Lemma 2.5.8 the maps $\phi_M(m)$ all belong to the image of $\theta_{M,\mathcal{A}_q}$. By Lemma 2.3.10 with $C = \mathcal{A}_q$, we conclude that M must be an $\mathcal{A}_q$-comodule. The same argument shows that integrable $U^{\mathrm{res}}(\mathfrak{b})$-modules are $\mathcal{B}_q$-comodules by Lemma 2.5.8 again.

Thus since the functors are fully faithful we are now reduced to showing that any $\mathcal{A}_q$-comodule (respectively $\mathcal{B}_q$-comodule) is integrable when viewed as a U^{res}-module (respectively $U^{\mathrm{res}}(\mathfrak{b})$-module). We prove it for $\mathcal{B}_q$, the proof for $\mathcal{A}_q$ being entirely analogous. Suppose M is a $\mathcal{B}_q$-comodule. Then by Remark 2.1.14, the map $\rho : M \to M \otimes_R \mathcal{B}_q$ is an injective comodule homomorphism where the right hand side is given the coaction map $1 \otimes \Delta$. In other words, in the language of $U^{\mathrm{res}}(\mathfrak{b})$-modules, this is saying the action on $M \otimes_R \mathcal{B}_q$ is the tensor product of the trivial action on M with the usual action on $\mathcal{B}_q$, i.e. for $u \in U^{\mathrm{res}}(\mathfrak{b})$ we have $u(m \otimes f) = m \otimes uf$ for all $m \in M$ and $f \in \mathcal{B}_q$. Thus, since $\mathcal{B}_q$ is integrable, so is $M \otimes_R \mathcal{B}_q$ with that structure. But now ρ identifies M with a submodule of $M \otimes_R \mathcal{B}_q$, hence it must be integrable by Lemma 2.4.15. □

Remark 2.5.13. Completely analogously, the category of $\mathcal{O}_q$-comodules, respectively $\mathcal{O}_q(B)$-comodules, is isomorphic to the category of integrable U_q-modules, respectively $U_q^{\geq 0}$-modules.

We now collect some facts about the induction functor for quantum groups from [4]. There is a functor $\mathrm{Ind} : \mathcal{B}_q\text{-comod} \to \mathcal{A}_q\text{-comod}$ defined as follows:

Definition 2.5.14. Given an integrable $U^{\mathrm{res}}(\mathfrak{b})$-module M, set

$$\mathrm{Ind}(M) = \{f \in H(M) : f \text{ is a } U^{\mathrm{res}}(\mathfrak{b})\text{-module homomorphism}\}.$$

This is a U^{res}-submodule of $H(M)$, hence integrable by Lemma 2.4.15. By [4, Section 2.10], $\mathrm{Ind}(M) \cong (M \otimes_R \mathcal{A}_q)^{U^{\mathrm{res}}(\mathfrak{b})}$ where the invariants are taken under the tensor $U^{\mathrm{res}}(\mathfrak{b})$-action on $M \otimes_R \mathcal{A}_q$.

This functor is by definition left exact, it satisfies Frobenius reciprocity meaning it is the right adjoint of restriction (see [4, Proposition 2.12]) and we can right derive it since the category of integrable $U^{\mathrm{res}}(\mathfrak{b})$-modules has enough injectives (see [4, Corollary 2.13] - we will prove this differently later in Lemma 4.2.2).

Now given $\lambda \in P$, there is a rank one integrable $U^{\mathrm{res}}(\mathfrak{b})$-module R_λ, which is just R equipped with the action defined by ψ_λ extended to $U^{\mathrm{res}}(\mathfrak{b})$. In other words $(U^{\mathrm{res}})^+$ acts as 0 and $(U^{\mathrm{res}})^0$ acts via the character ψ_λ. This is by definition integrable and so a $\mathcal{B}_q$-comodule. All the results we'll need are summarized in the following theorem:

Theorem 2.5.15. *Let M be an integrable $U^{res}(\mathfrak{b})$-module and let $\lambda \in P^+$.*

(i) ([4, Theorem 5.8(ii)]) *If M is finitely generated over R, so is $R^i\mathrm{Ind}(M)$ for any $i \geq 0$.*

(ii) ([4, Corollary 3.3]) *The U^{res}-module $\mathrm{Ind}(R_{-\lambda})$ is free over R and satisfies the Weyl character formula.*

(iii) (Kempf vanishing - see [4, Corollary 5.7]) *If M is finitely generated over R and $U^{res}(\mathfrak{b})$ acts on it via the character $\psi_{-\lambda}$, then $R^i\mathrm{Ind}(M) = 0$ for $i > 0$. In particular $R^i\mathrm{Ind}(R_{-\lambda}) = 0$ for $i > 0$.*

(iv) ([4, Theorem 5.8(iii)]) *$R^i\mathrm{Ind}(M) = 0$ for any $i > N$.*

Remark 2.5.16. Note that there is a sign change for λ between the results as stated above and the results as stated in [4]. This is because the results in *loc. cit.* are all stated for induction from the negative Borel subalgebra to the whole quantum group, while we follow the conventions from [13] and work with the positive Borel subalgebra instead. Hence, in our setting, $-\lambda$ is the *lowest* weight of $\mathrm{Ind}(R_{-\lambda})$.

Therefore we see from the above Theorem that each irreducible U_q-module $V(\lambda)$, $\lambda \in P^+$, contains a finite free R-submodule which is a U^{res}-module, namely $\mathrm{Ind}(R_{w_0\lambda})$. Analogously to the situation for $\mathcal{O}_q$, we have:

Proposition 2.5.17 ([4, Proposition & Remark 12.4])**.** *The algebra $\mathcal{A}_q$ is in fact generated by the matrix coefficients of the fundamental representations, i.e. by the matrix coefficients of Ind($R_{w_0\varpi_i}$) for $i = 1, \ldots, n$.*

We conclude by collecting a few facts about reduction modulo π in this setup. The key idea here is that, since $q^{\frac{1}{d}} \equiv 1 \pmod{\pi}$, we are in the situation of 'specialisation at $q = 1$' and all of our objects become non-quantum after reduction modulo π. We remark that some results in the literature about specialisation at at $q = 1$ are only stated over $\mathbb{C}$, but are completely general and work in our setup as well.

Proposition 2.5.18. *(i)* ([59, 8.15]) *The quotient k-algebra*

$$U_k^{res}/(K_{\varpi_1} - 1, \ldots, K_{\varpi_n} - 1)$$

is isomorphic to the hyperalgebra of the group G_k.

(ii) ([27, Proposition 9.2.3]) *The quotient k-algebra $U_k/(K_{\varpi_1} - 1, \ldots, K_{\varpi_n} - 1)$ is isomorphic to the enveloping algebra $U(\mathfrak{g}_k)$ of the k-Lie algebra $\mathfrak{g}_k$. In fact we have*

$$U_k \cong U(\mathfrak{g}_k)[K_\mu : \mu \in P]/(K_{\alpha_i}^2 - 1 : 1 \leq i \leq n)$$

as k-algebras.

(iii) ([4, 3.11 & Proposition 3.7]) *Suppose that M is an integrable $U^{res}(\mathfrak{b})$-module. Then so is M_k, and in fact M_k is a rational B_k-module, and we have an isomorphism between the cohomology $R^i\mathrm{Ind}(M_k) \cong R^i\mathrm{Ind}_{B_k}^{G_k}(M_k)$ for every $i \geq 0$, where the right hand side is the sheaf cohomology on the flag variety G_k/B_k. Moreover, if M_k is acyclic then we have an isomorphism $\mathrm{Ind}_{B_k}^{G_k}(M_k) \cong \mathrm{Ind}(M) \otimes_R k$.*

(iv) There is an isomorphism of k-algebras $\mathcal{A}_q \otimes_R k \cong \mathcal{O}(G_k)$.

Proof. We give a proof of (iv) here, leaving the rest to the literature. By Lemma 2.5.8, $\mathcal{A}_q \otimes_R k \cong H(k)$ and so identifies with the elements $f \in (U_k^{\mathrm{res}})^*$ which kill some ideal $I \subseteq U_k^{\mathrm{res}}$ which is cofinite and such that I contains a finite intersection of kernels of characters of the form ψ_λ. But note that since $q^{\frac{1}{d}} \equiv 1 \pmod{\pi}$, any character ψ_λ sends K_{ϖ_i} to 1 for any $1 \leq i \leq n$. Hence it follows that $K_{\varpi_i} - 1 \in I$ for each $1 \leq i \leq n$. Thus by (i), $\mathcal{A}_q \otimes_R k$ identifies with the matrix coefficients of finite dimensional modules over the hyperalgebra of G_k whose kernel contain a finite intersection of kernels of characters of the form ψ_λ. But finite dimensional modules over the hyperalgebra are all integrable, i.e. rational G_k-modules, by [28, 9.2] and hence all their matrix coefficients are of this form. Thus $\mathcal{A}_q \otimes_R k$ identifies in this way with the algebra of matrix coefficients of all finite dimensional rational G_k-modules, which is well-known to be $\mathcal{O}(G_k)$. □

2.6 Generalities on R-algebras and modules

We collect here some general facts on R-algebras and modules that we will need. We first establish a very elementary fact:

Lemma 2.6.1. *Suppose that A is a Noetherian R-algebra. Then $A_L = A \otimes_R L$ is Noetherian as well.*

Proof. Since the image of $A \to A_L$ is Noetherian, we may assume that A is a subring of A_L. Suppose that I is a left (resp. right) ideal in A_L. Then $I \cap A$ is a left (resp. right) ideal in A so is finitely generated. Since $A_L = A \otimes_R L$, I is generated by $I \cap A$ as a vector space and the result follows. □

We now establish some conditions under which we can lift the Noetherian property from the reduction mod π of a ring to the ring itself, or its π-adic completion.

Proposition 2.6.2. *(i) Suppose that A is an R-algebra such that $A/\pi A$ is Noetherian. Then the π-adic completion $\widehat{A}$ is also Noetherian.*

(ii) Let $n \geq 1$ and suppose that we have $\mathbb{Z}^n$-graded R-algebra $\mathcal{R} = \oplus_{\mathbf{m}\in\mathbb{Z}^n}\mathcal{R}_{\mathbf{m}}$ such that each graded piece $\mathcal{R}_{\mathbf{m}}$ is finitely generated over R. If $\mathcal{R}/\pi\mathcal{R}$ is Noetherian, then $\mathcal{R}$ is graded Noetherian.

Proof. (i) is just [17, Lemma 3.2.2]. For (ii) we use the same argument as in [56, Proposition II.2.3]. Specifically, consider the π-adic filtration on $\mathcal{R}$. The associated graded ring is a quotient of the polynomial algebra $(\mathcal{R}/\pi\mathcal{R})[t]$ (where t corresponds to the symbol of π), and so is Noetherian. We will consider several graded R-submodules of $\mathcal{R}$, equipped with the subspace filtration of the π-adic filtration.

Suppose we are given two graded ideals $I \subset J$ with $I \neq J$. Then we have $\operatorname{gr} I \subset \operatorname{gr} J$ and it will suffice to show that $\operatorname{gr} I \neq \operatorname{gr} J$. Pick $\mathbf{m} \in \mathbb{Z}^n$ such that $I_{\mathbf{m}} \neq J_{\mathbf{m}}$, and assume that $\operatorname{gr} I_{\mathbf{m}} = \operatorname{gr} J_{\mathbf{m}}$. Since $I_{\mathbf{m}}$ and $J_{\mathbf{m}}$ are finitely generated over R, we will get a contradiction by Nakayama if we show that $J_{\mathbf{m}} = I_{\mathbf{m}} + \pi J_{\mathbf{m}}$.

By the Artin-Rees Lemma ([10, Theorem 10.11]) applied to $J_{\mathbf{m}}$ viewed as a submodule of $\mathcal{R}_{\mathbf{m}}$, the subspace filtration of the π-adic filtration on $\mathcal{R}_{\mathbf{m}}$ and the π-adic filtration on $J_{\mathbf{m}}$ have finite difference. So there exists a $d \in \mathbb{Z}_{<0}$ such that for all $j \in J_{\mathbf{m}}$ with degree $d(j) < d$ in the subspace filtration, $j \in \pi J_{\mathbf{m}}$. Now let $j \in J_{\mathbf{m}}$ be arbitrary. We show by induction on $d(j)$ that $j \in I_{\mathbf{m}} + \pi J_{\mathbf{m}}$, the cases $d(j) < d$ being already dealt with. Since $\operatorname{gr} I_{\mathbf{m}} = \operatorname{gr} J_{\mathbf{m}}$, there exists $i \in I_{\mathbf{m}}$ such that $d(i-j) < d(j)$. But by induction hypothesis this implies $i-j \in I_{\mathbf{m}}+\pi J_{\mathbf{m}}$, and hence we get $j = i-(i-j) \in I_{\mathbf{m}}+\pi J_{\mathbf{m}}$ as required. □

Corollary 2.6.3. *The ring $\widehat{\mathcal{A}_q}$ is Noetherian.*

Proof. By Proposition 2.5.18(iv), the ring $\mathcal{A}_q/\pi\mathcal{A}_q$ coincides with the ring of regular functions on the group G_k and hence is Noetherian. Therefore the result follows from part (i) of the Proposition. □

We also have a useful criterion for determining flatness of π-adically complete algebras:

Lemma 2.6.4 ([17, 3.2.3(vii)])**.** *Suppose $f : A \to B$ is a ring homomorphism between two π-adically complete, Noetherian R-algebras. Then f makes B into a left (resp. right) flat A-module if the induced maps $A/\pi^n A \to B/\pi^n B$ make $B/\pi^n B$ into a left (resp. right) flat $A/\pi^n A$-module for all $n \geq 1$.*

We now turn to the notion of deformable algebras and modules.

Definition 2.6.5 ([6, Definition 3.5])**.** A positively $\mathbb{Z}$-filtered R-algebra A with F_0A an R-subalgebra of A is said to be a *deformable R-algebra* if $\operatorname{gr} A$ is a flat R-module and A is π-adically separated. A morphism between deformable R-algebras is a filtered R-algebra homomorphism. Analogously, given a filtered R-module M we say that M is deformable if $\operatorname{gr} M$ is a flat R-module and M is π-adically separated.

More generally, given any filtered R-module M, we may form its *n-th deformation* by taking the R-submodule

$$M_n = \sum_{i \geq 0} \pi^{ni} F_i M.$$

When $M = A$ is a filtered R-algebra, each deformation A_n is a subring.

Remark 2.6.6. Note that forcing deformable algebras to be π-adically separated is not a very big restriction, for instance it always holds when A is a Noetherian domain as long as π is not a unit by [55, Proposition I.4.4.5].

Lemma 2.6.7. *Let M be a deformable R-module. Then*

(i) ([6, Lemma 3.5]) *For all $n \geq 0$, M_n is also deformable, with filtration*

$$F_j M_n := M_n \cap F_j M = \sum_{i=0}^{j} \pi^{ni} F_i M,$$

and there is a natural isomorphism $gr\, M_n \cong gr\, M$.

(ii) ([8, Lemma 6.4(a)]) $M_1 \cap \pi^t M = \sum_{i \geq t} \pi^i F_i M$ *for any* $t \geq 0$*;*

(iii) ([8, Lemma 6.4(b)]) $(M_n)_m = M_{n+m}$ *for any* $n, m \geq 0$.

We also record here a useful fact about tensor products and deformations. Given two filtered R-modules M and N, we can give $M \otimes_R N$ the tensor filtration, where $F_t(M \otimes_R N)$ is generated as an R-module by all elementary tensors $m \otimes n$ such that $m \in F_i M$ and $n \in F_j N$ where $i + j = t$.

Lemma 2.6.8. *If M and N are torsion-free filtered R-modules, then $(M \otimes_R N)_n = M_n \otimes_R N_n$ for all $n \geq 0$.*

Proof. Since M and N are flat, we have an injective homomorphism $M_n \otimes_R N_n \to M \otimes_R N$. Identifying $M_n \otimes_R N_n$ with its image, we may assume that $M_n \otimes_R N_n$ and $(M \otimes_R N)_n$ both are submodules of $M \otimes_R N$. But now, for each $t \geq 0$, we have in $M \otimes_R N$ that $\pi^{tn}(a \otimes b) = \pi^{in} a \otimes \pi^{jn} b$, where $a \in F_i M$ and $b \in F_j N$ and $i + j = t$. Thus we see that $(M \otimes_R N)_n = M_n \otimes_R N_n$ since t was arbitrary. □

Hence $M \mapsto M_n$ is a monoidal endofunctor of the category of torsion-free filtered R-modules.

2.7 Basic nonarchimedean functional analysis

We now collect some standard facts from nonarchimedean functional analysis. For a more detailed treatment, see [73, 21]. We first begin with (semi-)norms, locally convex spaces, Banach spaces and Fréchet spaces. The field L is by definition equipped with a nonarchimedean absolute value, meaning that $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in L$, and $R = \{x \in L : |x| \leq 1\}$ is the unit ball of L with respect to it.

Definition 2.7.1. A (nonarchimedean) *semi-norm* on an L-vector space V is a function $q : V \to \mathbb{R}_{\geq 0}$ satisfying:

(i) $q(av) = |a|\, q(v)$ for all $v \in V$ and $a \in L$; and

(ii) $q(v + w) \leq \max(q(v), q(w))$ for all $v, w \in V$.

Condition (ii) is called the *strong triangle inequality.* Note that (i) implies in particular that $q(0) = 0$ and (i) and (ii) imply that the set $\{v \in V : q(v) = 0\}$ is a vector subspace of V. If furthermore the condition

(iii) $q(v) = 0$ if and only if $v = 0$

is satisfied then we say that q is a *norm.* When q is a norm, it is conventional to denote it by $||\cdot||$.

Given a family of semi-norms $(q_i)_{i\in I}$ on V, we may assign to V the coarsest topology such that $q_i : V \to \mathbb{R}$ is continuous for all $i \in I$ and all translation maps $v + \cdot : V \to V$, for $v \in V$, are continuous. In such a case we call V a *locally convex space.* If the topology is defined by a single (semi-)norm q, we call V a *(semi-)normed space.*

Notation. Given a semi-normed space V with semi-norm q, we denote its unit ball by $V^\circ := \{v \in V : q(v) \leq 1\}$. The strong triangle inequality ensures that V° is an R-submodule of V.

It is a well-known consequence of condition (ii) above that given an L-vector space V, $v, w \in V$ and a semi-norm q on V such that $q(v) \neq q(w)$, then $q(v+w) = \max(q(v), q(w))$ (see [73, Section 1, page 2]).

Lemma 2.7.2 ([73, Proposition 3.1])**.** *An L-linear map $f : V \to W$ between two semi-normed L-vector spaces is continuous if and only if it is bounded, i.e. there exists a real number $c \geq 0$ such that for all $v \in V$, $q_W(f(v)) \leq c \cdot q_V(v)$.*

Hence we see in particular that two norms $||\cdot||_1$ and $||\cdot||_2$ on V are equivalent if and only if the identity map $(V, ||\cdot||_1) \to (V, ||\cdot||_2)$ and its inverse are bounded. If we denote by V_1° and V_2° the unit balls of V with respect to $||\cdot||_1$ and $||\cdot||_2$ respectively, this says that

$$\pi^b V_2^\circ \subseteq V_1^\circ \subseteq \pi^a V_2^\circ$$

for some integers $a \leq b$.

We now turn to subspaces and quotients. If W is a vector subspace of V, we may equip W with the restriction of any (semi-)norm q on V, and we call that (semi-)norm on W the *subspace (semi-)norm.* Moreover we may define a *quotient (semi-)norm* on V/W by setting

$$q_{V/W}(v + W) := \inf_{w\in W} q(v + w).$$

The quotient semi-norm is a norm if and only if W is closed in V (see [21, Propositions 1.1.6/1 & 2.1.2/1]).

Given two normed spaces V and W, we may also define a norm on $V \oplus W$ by

$$||(v, w)|| = \max\{||v||_V, ||w||_W\}$$

for all $v \in V$ and all $w \in W$ (see [21, 2.1.5 Definition 1]).

Next, an R-submodule $M \subseteq V$ is called a *lattice* if $M \otimes_R L \cong V$. Now, given seminorms $q_1, \ldots, q_n$ on V and a positive real number $\epsilon > 0$, we may define a lattice in V by

$$V(q_1, \ldots, q_n; \epsilon) := \{v \in V : q_1(v), \ldots, q_n(v) \leq \epsilon\}.$$

By [73, Section 4, pages 13-14], the family $v + V(q_1, \ldots, q_n; \epsilon)$, for $v \in V$ and $\epsilon > 0$, forms the basis of a topology on V.

Lemma 2.7.3 ([73, Proposition 4.3])**.** *The topology on a locally convex space V coincides with the topology defined by the family of lattices $\{V(q_{i_1}, \ldots, q_{i_n}; \epsilon) : i_1, \ldots, i_n \in I, \epsilon > 0\}$ as above.*

Hence we see that if $||\cdot||$ is a norm on V, the normed space topology on V is the usual metric topology coming from $||\cdot||$.

Definition 2.7.4. A normed space which is complete, meaning that all Cauchy sequences converge, is called a *Banach space.* More generally, a locally convex space which is metrizable and complete is called a *Fréchet space.*

Of course, any Banach space is in particular a Fréchet space. Also, L is itself a Banach space with the absolute value as a norm.

Lemma 2.7.5. *(i)* ([73, Proposition 8.3]) *Let V be a Fréchet (resp. Banach) space and let $W \subseteq V$ be a closed vector subspace. Then V/W with the quotient topology is a Fréchet (resp. Banach) space.*

(ii) ([21, 2.1.5 Proposition 6]) *If V and W are Banach spaces, then the norm on the direct sum $V \oplus W$ makes it into a Banach space.*

When a locally convex space is finite dimensional, it has nice properties:

Proposition 2.7.6 ([73, Proposition 4.13])**.** *Every Hausdorff locally convex topology on a finite dimensional vector space $V = L^n$ $(n \geq 1)$ is equivalent to the one defined by the norm $||(a_1, \ldots, a_n)|| = \max_{1 \leq i \leq n} |a_i|$.*

Hence we see from Lemma 2.7.5(ii) that every finite dimensional Hausdorff locally convex space is in fact a Banach space, and there is a unique such topology.

We now turn to the set of all bounded linear maps $\mathrm{Hom}_L(V, W)$ between two normed spaces V and W. It has the structure of a normed space, given by the *operator norm*

$$||f|| = \sup\left\{\frac{||f(v)||}{||v||} : v \neq 0\right\}$$

(note we are implicitly assuming that $V \neq 0$). When W is a Banach space, so is $\mathrm{Hom}_L(V, W)$ (see [73, Proposition 3.3]).

Next, we turn to the notion of completeness for locally convex spaces. A family $(v_i)_{i\in I}$ of vectors in locally convex space V such that I is a directed set is called a *net.* Say that the topology on V is defined by a family $(q_j)_{j\in J}$ of seminorms. A net is said to be Cauchy if for any $j_1, \dots, j_n \in J$ and $\epsilon > 0$, there is an index $i \in I$ such that $v_j - v_k \in V(q_{j_1}, \dots, q_{j_n}; \epsilon)$ for all $j, k \geq i$. Then we say that V is *complete* if every Cauchy net converges, i.e. if given a Cauchy net $(v_i)_{i\in I}$, there is a vector $v \in V$ such that $(q_j(v_i - v))_{i\in I}$ tends to zero for every $j \in J$. If V is a normed space (or more generally metrizable), this agrees with the usual (metric) notion of completeness (see [73, Remark 7.2]).

Proposition 2.7.7 ([73, Proposition 7.3])**.** *For any locally convex space V there exists a unique, up to topological isomorphism, complete Hausdorff locally convex space $\widehat{V}$ and a continuous linear map $c_V : V \to \widehat{V}$ satisfying the following universal property: given any continuous linear map $f : V \to W$ where W is a complete Hausdorff locally convex space, there is a unique continuous linear map $\widehat{f} : \widehat{V} \to W$ such that $f = \widehat{f} \circ c_V$. Moreover, the map c_V has dense image and it induces a topological isomorphism between $V/\overline{\{0\}}$ with the quotient topology and $\mathrm{Im}(c_V)$ with the subspace topology.*

When working with normed spaces, the completion can be described more explicitly by:

Proposition 2.7.8 ([18, Theorem 2.11])**.** *If V is a normed space with unit ball V°, then there is a canonical isomorphism $\widehat{V} \cong \widehat{V^\circ} \otimes_R L$.*

Definition 2.7.9. The locally convex space $\widehat{V}$ is called the *Hausdorff completion* of V.

We now seek to describe Fréchet spaces more explicitly. To this end, we have the following result:

Proposition 2.7.10 ([73, Proposition 8.1])**.** *A Hausdorff locally convex space V is metrizable if and only if there is a countable increasing family of seminorms $q_1 \leq q_2 \leq \cdots$ which define the topology on V.*

We note that if V is a semi-normed space with semi-norm q, then q induces a norm on $V/\{v \in V : q(v) = 0\}$ and $\widehat{V}$ coincides with the metric space completion of it and is a Banach space.

Notation. We will write V_q for the above Banach space completion of a vector space V with respect to a semi-norm q.

Now if V is a Fréchet space with seminorms $q_1 \leq q_2 \leq \cdots$ defining its topology as above, we have for each $i \geq 1$ that the identity map on V extends to a continuous map $V_{q_{i+1}} \to V_{q_i}$ with dense image such that the diagram

$$\begin{tikzcd}
& V_{q_{i+1}} \arrow[dd] \\
V \arrow[ur] \arrow[dr] & \\
& V_{q_i}
\end{tikzcd}$$

commutes. Thus we may form the inverse limit $\varprojlim V_{q_i}$, equipped with the coarsest topology such that the natural maps $f_j : \varprojlim V_{q_i} \to V_{q_j}$, for $j \geq 1$, are all continuous. In other words we equip it with the locally convex topology given by the semi-norms $q_j \circ f_j$. Moreover, there is a continuous map

$$V \to \varprojlim V_{q_i}.$$

Since V is complete and its topology is by definition the coarsest topology making the maps $V \to V_{q_i}$, for $i \geq 1$, continuous, we see that this map is in fact a topological isomorphism (see more generally [69, Theorem I.6.1 page 28 & Chapter II.4, page 48f]).

The next result, called the Open Mapping Theorem, is very well-known and we will use it several times.

Proposition 2.7.11 ([73, Proposition 8.6])**.** *Any continuous surjection $f : V \to W$ between Fréchet spaces is an open map.*

Corollary 2.7.12. *Any continuous linear bijection between two Fréchet space is a topological isomorphism.*

Next, we describe how a lattice inside an L-vector space V may be used to define a semi-norm on V.

Definition 2.7.13 ([73, Section 2, page 8])**.** Given a lattice $M \subset V$, we may define a semi-norm on V, called the *gauge semi-norm*, given by

$$q_M(v) = \inf_{\substack{a \in L \\ v \in aM}} |a| .$$

Note that this infimum, if it is non-zero, equals $|\pi^n|$ where $n \in \mathbb{Z}$ is the largest integer such that $v \in \pi^n M$.

Hence we see that the topology induced by the gauge norm is the topology induced by the π-adic filtration on M. Thus we see that the gauge semi-norm is a norm if and only if M is π-adically separated. In that case, the unit ball of V with respect to this norm is M and we will denote the gauge norm by $||\cdot||_M$. Moreover, V is then Banach if and only if M is π-adically complete. This gives:

Lemma 2.7.14. *Suppose that M is a torsion-free, π-adically complete R-module. Then $M_L = M \otimes_R L$ is a Banach space with respect to the gauge norm associated to M.*

It is clear from the axioms of a norm that if V is a normed space then V° is π-adically separated: if $v \in \cap_{n \geq 0} \pi^n V^\circ$ then $||v|| \leq |\pi^n|$ for every $n \geq 0$ and so $||v|| = 0$, i.e. $v = 0$. We then have:

Lemma 2.7.15 ([73, Lemma 2.2])**.** *Suppose that V is a normed space with norm $||\cdot||$ and unit ball V°. Then $||\cdot||$ is equivalent to the gauge norm $||\cdot||_{V^\circ}$.*

We now see how gauge norms behave with respect to subspaces and quotients.

Lemma 2.7.16. *Suppose that V is a normed space with respect to the gauge norm of a π-adically separated lattice M and let W be a closed vector subspace of V. Then the unit ball of V/W with respect to the quotient norm is equal to the image of M in the quotient.*

Proof. By definition of the quotient norm, the image of M is contained in $(V/W)^\circ$. Conversely, let $v \in V$ and assume that $\lambda := ||v + W||_{V/W} \leq 1$. By definition of the quotient norm, if $\lambda < 1$ then there is a $w \in W$ such that $||v + w||_M < 1$ and $v + w \in M$ as required. If $\lambda = 1$, then for any $\epsilon > 0$, there is some $w \in W$ such that $||v + w||_M < 1 + \epsilon$. So the largest $a \in \mathbb{Z}$ such that $v + w \in \pi^a M$ satisfies $|\pi|^a < 1 + \epsilon$. By discreteness of the set $|\pi|^{\mathbb{Z}}$, this implies that for ϵ small enough we have $v + w \in M$ as required. □

Thus we see from the previous two Lemmas that if V is a normed space, W is a closed vector subspace and $p : V \to V/W$ is the projection map, then the quotient norm on V/W is equivalent to the gauge norm of $p(V^\circ)$.

Lemma 2.7.17. *Suppose that V is a semi-normed space with respect to the gauge norm q_M of a lattice M and let W be a vector subspace. Then the restriction of q_M to W is $q_{M\cap W}$.*

Proof. For $w \in W$ and $a \in L$, we have $w \in aM$ if and only if $w \in a(M \cap W)$. □

We conclude our general review of nonarchimedean functional analysis with the notion of strictness. Given two semi-normed spaces V and W and a continuous linear map $f : V \to W$, there is an induced continuous linear bijection

$$\bar{f} : V/\ker(f) \to \mathrm{Im}(f)$$

where the left hand side is given the quotient topology, and the right hand side is given the subspace topology. However, this may not be a topological isomorphism, and so is not an isomorphism in the category of semi-normed spaces. Hence we see that the category of semi-normed spaces is not abelian (and the same can be said of locally convex spaces, normed spaces or Fréchet/Banach spaces). This motivates:

Definition 2.7.18. A continuous linear map $f : V \to W$ between two semi-normed spaces is called *strict* if the induced map $\bar{f} : V/\ker(f) \to \mathrm{Im}(f)$ is a topological isomorphism.

Strict maps behave well with respect to the completion functor we introduced earlier:

Proposition 2.7.19. *Let V, W and X be semi-normed spaces.*

(i) ([21, 1.1.9/4-6]) *If $f : V \to W$ is a strict continuous linear map, then the induced map $\widehat{f} : \widehat{V} \to \widehat{W}$ is also strict. Moreover there are canonical isomorphisms* $\ker(\widehat{f}) \cong \widehat{\ker(f)}$ *and* $\mathrm{Im}(\widehat{f}) \cong \widehat{\mathrm{Im}(f)}$. *Hence if f is injective, respectively surjective, then so is $\widehat{f}$.*

(ii) ([21, 1.1.9 Corollary 6]) *Suppose $V \xrightarrow{f} W \xrightarrow{g} X$ is a sequence of semi-normed spaces with strict maps, which is exact as a sequence of L-vector spaces. Then $\widehat{V} \xrightarrow{\widehat{f}} \widehat{W} \xrightarrow{\widehat{g}} \widehat{X}$ is also exact as a sequence of L-vector spaces.*

We now summarize a few more basic facts about strict maps which will be useful:

Lemma 2.7.20. *Let V, W and X be normed spaces, and let $f : V \to W$ and $g : W \to X$ be continuous linear maps.*

(i) ([21, 1.1.9 Lemma 2]) *The map f is strict if and only if there exists a real constant $c > 0$ such that for all $v \in V$, $||v||_V \leq c \cdot ||f(v)||_W$.*

(ii) If f and g are strict and injective, then so is $g \circ f$.

(iii) If f is a split injection, i.e. there is a continuous linear map $h : W \to V$ such that $h \circ f = \mathrm{id}_V$, then f is strict.

Proof. We prove (ii) and (iii). For (ii), by definition of strict, f and g define topological isomorphisms onto their image. Equivalently, there are real numbers $C_1, C_2, D_1, D_2 > 0$ such that for all $v \in V$ and all $w \in W$, we have

$$C_1 \, ||v||_V \leq ||f(v)||_W \leq C_2 \, ||v||_V \quad \text{and} \quad D_1 \, ||w||_W \leq ||g(w)||_X \leq D_2 \, ||w||_W \, .$$

But by putting $w = f(v)$, this gives $C_1 D_1 \, ||v||_V \leq ||g(f(v))||_X \leq C_2 D_2 \, ||v||_X$, so that $g \circ f$ also defines a homeomorphism onto its image, and is therefore strict.

For (iii), we have

$$||v||_V = ||h(f(v))||_V \leq ||h|| \, ||f(v)||_W$$

for all $v \in V$. Hence the result follows by (i). □

2.8 Banach algebras, completed tensor products and Hopf algebras

In this Section, we introduce Banach and Fréchet algebras, semi-norms on tensor products and use this to introduce the concept of Hopf algebra objects in the categories of Banach and Fréchet spaces.

Definition 2.8.1. An L-algebra A which is equipped with a norm $||\cdot||$ is called a *normed algebra* if

$$||1|| = 1, \text{ and } ||ab|| \leq ||a|| \, ||b||$$

for all $a, b \in A$. If the norm makes A into a Banach space, we say that A is a *Banach algebra.*

An A-module $\mathcal{M}$ equipped with a norm is called a *normed module* if

$$||am||_{\mathcal{M}} \leq ||a||_A \, ||m||_{\mathcal{M}}$$

for all $a \in A$ and all $m \in \mathcal{M}$. In other words, the A-action gives rise to a continuous linear map $A \to \mathrm{Hom}_L(\mathcal{M}, \mathcal{M})$ of norm at most 1. If the norm makes $\mathcal{M}$ into a Banach space, we call $\mathcal{M}$ a *Banach module.*

If A is a normed algebra and $\mathcal{M}$ is a normed A-module, then the A-action on $\mathcal{M}$ is continuous and so given any A-submodule $\mathcal{N}$ of $\mathcal{M}$, the closure $\overline{\mathcal{N}}$ is also an A-submodule. Moreover, it follows directly from the definition that A° is an R-subalgebra of A and $\mathcal{M}^\circ$ is an A°-submodule of $\mathcal{M}$. In fact we have:

Lemma 2.8.2. *Let A and $\mathcal{M}$ be normed spaces such that A is an L-algebra and $\mathcal{M}$ is an A-module. Then A is a normed algebra if and only if A° is an R-subalgebra of A. If these conditions hold, then $\mathcal{M}$ is a normed A-module if and only if $\mathcal{M}^\circ$ is an A°-submodule of $\mathcal{M}$.*

Proof. We have already seen one direction. Suppose first that $\mathcal{M}^\circ$ is an A°-submodule of $\mathcal{M}$, without assuming A is a normed algebra. Then for all $0 \neq a \in A$ and $0 \neq m \in \mathcal{M}$, we have that

$$\frac{a}{||a||_A} \cdot \frac{m}{||m||_M} \in \mathcal{M}^\circ,$$

and so after taking norms we get $||am||_{\mathcal{M}} \leq ||a||_A \, ||m||_{\mathcal{M}}$. From this, the second part of the statement is immediate, and the first part mostly follows by taking $\mathcal{M} = A$ with A-action given by left multiplication. The only thing left to check there is that $||1||_A = 1$. As A° is an R-subalgebra, we have $||1||_A \leq 1$. But from the above we also have

$$||1||_A = ||1 \cdot 1||_A \leq ||1||_A \cdot ||1||_A$$

which implies that $||1||_A \geq 1$ as required. □

Corollary 2.8.3. *Let A be a normed algebra and $\mathcal{M}$ a normed A-module. If we equip A with the gauge norm associated with A°, then A is a normed algebra with this new norm. Moreover, with this new norm on A, the gauge norm on $\mathcal{M}$ associated with $\mathcal{M}^\circ$ makes $\mathcal{M}$ into a normed A-module.*

We will also need the analogous notion of a *Fréchet algebra*. It is defined to be an L-algebra A which is also a Fréchet space such that the two structures are compatible as follows: if $q_1 \leq q_2 \leq \cdots$ is a family of seminorms defining the topology on A, then we require that each q_i is an *algebra seminorm*, meaning that $q_i(ab) \leq q_i(a)q_i(b)$ for all $a, b \in A$. Note that the canonical topological isomorphism $A \to \varprojlim A_{q_i}$ is then also an algebra isomorphism.

Remark 2.8.4. Some authors have a slightly weaker definition of Banach (resp. Fréchet) algebras: they require more generally that there is a constant $c > 0$ such that $||ab|| \leq c \cdot ||a|| \, ||b||$ (resp. $q_i(ab) \leq c \cdot q_i(a)q_i(b)$).

Notation. Given a Banach algebra A, we denote by $\mathbf{Mod}(A)$ the category of Banach A-modules, with morphisms given by continuous (i.e. bounded) module maps.

Of course, L itself is a Banach algebra. Also for a normed algebra A, any finite direct sum $A^{\oplus r}$ is a normed A-module.

Lemma 2.8.5. *Suppose that B is a torsion-free R-algebra such that B_k is Noetherian. Then $\widehat{B_L} := \widehat{B} \otimes_R L$ is a Noetherian Banach algebra.*

Proof. The Noetherian property follows immediately from Proposition 2.6.2(i) and Lemma 2.6.1. The Banach property is just Lemma 2.7.14. We note that the gauge norm on $\widehat{B_L}$ associated to $\widehat{B}$ makes $\widehat{B_L}$ into a normed algebra by Lemma 2.8.2. □

Now if A is a Banach algebra and $\mathcal{M}$ is any finitely generated A-module, then there is by definition a surjection $f : A^{\oplus r} \to \mathcal{M}$ for some $r \geq 1$. We may then equip $\mathcal{M}$ with the gauge semi-norm with respect to $N = f((A^\circ)^{\oplus r})$. We then have:

Lemma 2.8.6 ([21, Propositions 3.7.2/2 & 3.7.3/2-3])**.** *Suppose A is a Noetherian Banach algebra. Then a Banach A-module is finitely generated if and only if all of its submodules are closed. Moreover, any finitely generated A-module has a unique, up to equivalence, Banach A-module structure, which is given by the above norm. All module maps between finitely generated A-modules are then continuous and strict with respect to this Banach topology.*

We will be interested in the situation where not only the Banach algebra A is Noetherian, but in fact its unit ball A° is Noetherian as well (this of course implies that A is Noetherian by Lemma 2.6.1).

Corollary 2.8.7. *If A is a Banach algebra such that A° is Noetherian, and $\mathcal{M}$ is a finitely generated Banach A-module, then $\mathcal{M}^\circ$ is finitely generated over A°.*

Proof. By Lemma 2.8.6, the norm on $\mathcal{M}$ is equivalent to the quotient norm coming from a surjection $f : A^{\oplus r} \to \mathcal{M}$. If we let $N = f((A^\circ)^{\oplus r})$, this means in particular that there is some $a \in \mathbb{Z}$ such that

$$\mathcal{M}^\circ \subseteq \pi^a N.$$

As N is finitely generated and A° is Noetherian, the result follows. □

Having established the notion of algebras, we now turn to coalgebras. To that end, we need to introduce topologies on tensor products.

Definition 2.8.8 ([73, Section 17B, page 103])**.** Let V and W be two L-vector spaces, and let p and q be two semi-norms on V and W respectively. The *tensor product seminorm* $p \otimes q$ on $V \otimes_L W$ is defined in the following way: for $x \in V \otimes_L W$, we have

$$p \otimes q(x) := \inf \Big\{ \max_{1 \leq i \leq r} p(v_i) \cdot q(w_i) : x = \sum_{i=1}^{r} v_i \otimes w_i, v_i \in V, w_i \in W \Big\}.$$

If V and W are locally convex spaces with topologies defined by the families of semi-norms $(p_i)_{i \in I}$ and $(q_j)_{j \in J}$ respectively, then $V \otimes_L W$ may be then be given the locally convex topology given by the family of semi-norms $(p_i \otimes q_j)_{(i,j) \in I \times J}$. This is called the *projective tensor product topology.* One can then construct the Hausdorff completion $V \widehat{\otimes}_L W$ of this space, which we call the *completed tensor product* of V and W.

Note that this construction is functorial, so that two continuous linear maps $f : V \to W$ and $g : X \to Y$ induce a continuous linear map $f \widehat{\otimes} g : V \widehat{\otimes}_L X \to W \widehat{\otimes}_L Y$.

Since torsion-free R-modules are flat, if M and N are lattices in V and W respectively, then there is a canonical embedding

$$M \otimes_R N \to V \otimes_R W = V \otimes_L W$$

so that we may view $M \otimes_R N$ as a lattice inside $V \otimes_L W$. We can now gather a few facts about tensor product semi-norms:

Proposition 2.8.9. *Let V, W, M, N, p and q be as in the above. Then:*

(i) ([73, Proposition 17.4(ii)]) *$p \otimes q$ is a norm if and only if p and q are norms;*

(ii) ([73, Lemma 17.2]) *if $p = q_M$ and $q = q_N$ are gauge semi-norms, then $p \otimes q = q_{M \otimes_R N}$ is the gauge semi-norm with respect to the lattice $M \otimes_R N \subset V \otimes_L W$; and*

(iii) ([73, Proposition 17.4(iii)]) *if $V_0 \subset V$ and $W_0 \subset W$ are vector subspaces, then $p|_{V_0} \otimes q|_{W_0} = (p \otimes q)|_{V_0 \otimes_L W_0}$.*

By using Proposition 2.7.19(i), part (iii) of the above Proposition immediately implies:

Corollary 2.8.10 ([73, Corollary 17.5(ii)])**.** *If V and W are locally convex spaces and $V_0 \subset V$ and $W_0 \subset W$ are vector subspaces equipped with the subspace topology, then the projective tensor product topology on $V_0 \otimes_L W_0$ coincides with the subspace topology induced from $V \otimes_L W$. In other words, the injective continuous linear map $V_0 \otimes_L W_0 \to V \otimes_L W$ is strict. Hence the map $V_0 \widehat{\otimes}_L W_0 \to V \widehat{\otimes}_L W$ is a strict injection.*

The projective tensor product is in fact very well-behaved with respect to strictness:

Lemma 2.8.11 ([16, Appendix A, Lemma A.34])**.** *Suppose V, W and X are normed spaces, and let $f : V \to W$ be a strict surjective continuous linear map. Then the induced maps $f \otimes \mathrm{id}_X : V \otimes_L X \to W \otimes_L X$ and $f \widehat{\otimes} \mathrm{id}_X : V \widehat{\otimes}_L X \to W \widehat{\otimes}_L X$ are also both strict surjections.*

We see from Proposition 2.8.9(i) that the tensor product of two normed spaces V and W is itself a normed space via the tensor product norm, and it follows that $V \widehat{\otimes}_L W$ is then a Banach space. More generally, if V and W are Fréchet spaces, then so is $V \widehat{\otimes}_L W$ (see [73, Chapter 17, page 107]). Hence we see that $\widehat{\otimes}_L$ is a monoidal structure on the categories of Banach and Fréchet spaces. We also note that by definition, the multiplication on a Banach or Fréchet algebra A induces a continuous map $A \otimes_L A \to A$ and so a continuous map $A \widehat{\otimes}_L A \to A$ by completeness of A.

We now describe the completed tensor product of two Fréchet spaces more specifically. Suppose that $p_1 \leq p_2 \leq \cdots$ and $q_1 \leq q_2 \leq \cdots$ are countable increasing families of semi-norms defining the topologies on V and W respectively. Then $V \cong \varprojlim V_{p_i}$ and $W \cong \varprojlim W_{q_i}$, meaning that V and W are determined by the maps $V \to V_{p_i}$ and $W \to W_{q_i}$, for $i \geq 1$. Now, by definition the topology on $V \otimes_L W$ is determined by the maps $f_{ij} : V \otimes_L W \to V_{p_i} \otimes_L W_{q_j}$, for $i, j \geq 1$. But for any $1 \leq i \leq j$, we have a commutative diagram

$$\begin{tikzcd}
 & V_{p_j} \otimes_L W_{q_j} \arrow[d] \\
V \otimes_L W \arrow[ur, "f_{jj}"] \arrow[dr, "f_{ij}"'] & \\
 & V_{p_i} \otimes_L W_{q_j}
\end{tikzcd}$$

so that we see that the map f_{ij} is automatically continuous if f_{jj} is continuous. Thus the topology on $V \otimes_L W$ is determined by the maps f_{ii}, for $i \geq 1$. In fact we have:

Proposition 2.8.12 ([36, Proposition 1.1.29]). *Let V and W be L-Fréchet spaces whose topologes are defined by families of seminorms $p_1 \leq p_2 \leq \ldots \leq p_n \leq \ldots$ and $q_1 \leq q_2 \leq \ldots \leq q_n \leq \ldots$ respectively. Then we have a canonical isomorphism of Fréchet spaces*

$$V \widehat{\otimes}_L W \cong \varprojlim V_{p_n} \widehat{\otimes}_L W_{q_n}.$$

When V and W are Fréchet algebras and all the seminorms are algebra seminorms, this is an algebra isomorphism.

We will sometimes require to work more generally with topologies on tensor products of Banach modules over a Banach algebra, so we quickly discuss what we'll need here:

Definition 2.8.13 ([21, 2.1.7]). Let A be a Banach algebra, $\mathcal{M}$ a Banach right A-module, $\mathcal{N}$ a Banach left A-module. The completed tensor product $\mathcal{M} \widehat{\otimes}_A \mathcal{N}$ is defined to be the completion of the tensor product $\mathcal{M} \otimes_A \mathcal{N}$ with respect to the *tensor semi-norm*:

$$q(x) := \inf \Big\{ \max_{1 \leq i \leq r} ||m_i||_{\mathcal{M}} \cdot ||n_i||_{\mathcal{N}} : x = \sum_{i=1}^{r} m_i \otimes n_i, m_i \in \mathcal{M}, n_i \in \mathcal{N} \Big\}$$

for $x \in \mathcal{M} \otimes_A \mathcal{N}$.

When $A = L$, this of course agrees with the tensor product topology we've been describing. In this more general setup, a lot of the results we mentioned before, especially the ones to do with strictness of maps, do *not* hold. However, we still have the following generalisation of Proposition 2.8.9(ii):

Lemma 2.8.14 ([19, Lemma 2.2 & Corollary 2.3]). *Suppose that A, $\mathcal{M}$ and $\mathcal{N}$ are as above. If we equip them with the gauge norms from their unit ball, then the unit ball of the corresponding tensor semi-norm $\mathcal{M} \otimes_A \mathcal{N}$ is equal to the image of the natural map $\mathcal{M}^\circ \otimes_{A^\circ} \mathcal{N}^\circ \to \mathcal{M} \otimes_A \mathcal{N}$. Hence more generally, the semi-norm on $\mathcal{M} \otimes_A \mathcal{N}$ is equivalent to the gauge semi-norm associated with the image of $\mathcal{M}^\circ \otimes_{A^\circ} \mathcal{N}^\circ \to \mathcal{M} \otimes_A \mathcal{N}$.*

Now suppose that B is another Banach algebra, and assume there is a continuous algebra homomorphism $f : A \to B$. Then this induces a functor $f^* : \mathbf{Mod}(A) \to \mathbf{Mod}(B)$ given by $\mathcal{M} \mapsto B \widehat{\otimes}_A \mathcal{M}$. We will investigate this functor later, but we record here the following consequence of Lemma 2.8.14:

Corollary 2.8.15. *Suppose that A and B are Banach algebras and let $f : A \to B$ be a continuous algebra homomorphism. Assume that $f(A^\circ) \subseteq B^\circ$ and that B° is flat as a right A°-module. Let $\mathcal{M}$ be a π-adically complete A°-module which is π-torsion free. Then we have an isomorphism*

$$\widehat{B^\circ \otimes_{A^\circ} \mathcal{M}} \otimes_R L \cong B \widehat{\otimes}_A \mathcal{M}_L$$

and the B°-module $B^\circ \otimes_{A^\circ} \mathcal{M}$ is π-torsion free.

Proof. Since $\mathcal{M} \to \mathcal{M}_L$ is injective, by flatness we have an injection

$$B^\circ \otimes_{A^\circ} \mathcal{M} \hookrightarrow B^\circ \otimes_{A^\circ} \mathcal{M}_L \cong B \otimes_A \mathcal{M}_L.$$

Thus we immediately get that $B^\circ \otimes_{A^\circ} \mathcal{M}$ is π-torsion free, and the required isomorphism is obtained by taking completions and applying Lemma 2.8.14 and Proposition 2.7.8. □

As promised, we now define coalgebras and Hopf algebras in the categories of Banach and Fréchet spaces:

Definition 2.8.16. An *Banach coalgebra* (resp. *Fréchet coalgebra*) is a coalgebra object in the monoidal category of Banach spaces (resp, Fréchet spaces). In other words it is a Banach (resp. Fréchet) space C equipped with continuous linear maps $\Delta : C \to C \widehat{\otimes}_L C$ and $\varepsilon : C \to L$ which satisfy the usual axioms:

$$(\Delta \widehat{\otimes} \operatorname{id}_C) \circ \Delta = (\operatorname{id}_C \widehat{\otimes} \Delta) \circ \Delta, \quad (\operatorname{id}_C \widehat{\otimes} \varepsilon) \circ \Delta = (\varepsilon \widehat{\otimes} \operatorname{id}_C) \circ \Delta = \operatorname{id}_C .$$

A morphism of coalgebras $f : C \to D$ is then a continuous linear map such that $\varepsilon_D \circ f = \varepsilon_C$ and $(f \widehat{\otimes} f) \circ \Delta_C = \Delta_D \circ f$.

Given a Banach coalgebra C as above, a *right Banach C-comodule* is a Banach space $\mathcal{M}$ equipped with a continuous linear map $\rho_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \widehat{\otimes}_L C$, which satisfies:

$$(\operatorname{id}_{\mathcal{M}} \widehat{\otimes} \Delta) \circ \rho_{\mathcal{M}} = (\rho_{\mathcal{M}} \widehat{\otimes} \operatorname{id}_C) \circ \rho_{\mathcal{M}}, \quad (\operatorname{id}_{\mathcal{M}} \widehat{\otimes} \varepsilon) \circ \rho_{\mathcal{M}} = \operatorname{id}_{\mathcal{M}} .$$

A morphism of comodules $f : \mathcal{M} \to \mathcal{N}$ is then a continous linear map such that $\rho_{\mathcal{N}} \circ f = \rho_{\mathcal{M}} \circ (f \widehat{\otimes} \operatorname{id}_C)$. We denote by $\mathbf{Comod}(C)$ the category of right Banach C-comodules.

A *Banach Hopf algebra* (resp. *Fréchet Hopf algebra*) is a Banach (resp. Fréchet) algebra H which is also a coalgebra as above, such that Δ and ε are algebra homomorphisms, and furthemore H is equipped with a continuous linear map $S : H \to H$, which satisfies

$$m \circ (S \widehat{\otimes} \operatorname{id}_H) \circ \Delta = \iota \circ \varepsilon = m \circ (\operatorname{id}_H \widehat{\otimes} S) \circ \Delta$$

where $m : H \widehat{\otimes}_L H \to H$ and $\iota : L \to H$ denote the multiplication map and the unit in H respectively. A morphism of Hopf algebras $f : H \to G$ is then a continuous algebra homomorphism which is also a morphism of coalgebras, such that $S_G \circ f = f \circ S_H$.

Convention. We will only ever consider right Banach comodules, and so we will often say ‘comodule’ to mean ‘right comodule’.

We now quickly describe coideals and subcomodules in this setup. Suppose first that C is a Banach coalgebra, and let $I \subseteq C$ be a closed subspace equipped with the subspace norm. Then by Corollary 2.8.10, the canonical maps $I\widehat{\otimes}_L C \to C\widehat{\otimes}_L C$ and $C\widehat{\otimes}_L I \to C\widehat{\otimes}_L C$ are strict injections, so that we may identify $C\widehat{\otimes}_L I$ and $I\widehat{\otimes}_L C$ as subspaces of $C\widehat{\otimes}_L C$ with the subspace norm. Then we have:

Lemma 2.8.17. *With C and I as above, the kernel of the canonical map $C\widehat{\otimes}_L C \to C/I\widehat{\otimes}_L C/I$ is $C\widehat{\otimes}_L I + I\widehat{\otimes}_L C$.*

Proof. By the same argument as in the proof of Lemma 2.1.3, we have a commutative diagram

$$\begin{tikzcd}
 & C \otimes_L I \arrow[r] \arrow[d] & C/I \otimes_L I \arrow[r] \arrow[d] & 0 \\
I \otimes_L C \arrow[r] & C \otimes_L C \arrow[r] \arrow[dr, "\theta"] & C/I \otimes_L C \arrow[r] \arrow[d] & 0 \\
 & & C/I \otimes_L C/I \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

where all the rows and columns are exact. But now all the maps in the diagram are also strict by Corollary 2.8.10 and Lemma 2.8.11. Taking completion, we then obtain the same diagram, replacing every tensor product by its completion, where the rows and columns are still exact by Proposition 2.7.19(ii). The result now follows by a simple diagram chase. ☐

Hence we say that I is a *coideal* if $\Delta(I) \subseteq C\widehat{\otimes}_L I + I\widehat{\otimes}_L C$ and if $\varepsilon(I) = 0$. That way, the quotient C/I naturally inherits the structure of a Banach coalgebra from C.

Similarly, given a Banach C-comodule $\mathcal{M}$ with coaction ρ and a closed subspace $\mathcal{N}$, we say that $\mathcal{N}$ is a *subcomodule* of $\mathcal{M}$ if $\rho(\mathcal{N}) \subseteq \mathcal{N}\widehat{\otimes}_L C$. This ensures the restriction of ρ to $\mathcal{N}$ naturally makes $\mathcal{N}$ into a Banach C-comodule, and also induces a Banach C-comodule to the quotient $\mathcal{M}/\mathcal{N}$. We then have:

Lemma 2.8.18 ([32, Lemma II.1.1])**.** *Suppose that $f : \mathcal{M} \to \mathcal{N}$ is a morphism of C-comodules. Then* $\ker(f)$ *and* $\overline{\mathrm{Im}(f)}$ *are subcomodules of $\mathcal{M}$ and $\mathcal{N}$ respectively.*

We may also consider a completed tensor product $\mathcal{M}\widehat{\otimes}_L\mathcal{N}$ of two Banach H-comodule over a Banach Hopf algebra H, equipped with a tensor comodule structure. The definition of this structure is completely analogous to the classical tensor product of comodules:

$$\mathcal{M}\widehat{\otimes}_L\mathcal{N} \xrightarrow{\rho_{\mathcal{M}}\otimes\rho_{\mathcal{N}n}} \mathcal{M}\widehat{\otimes}_L H\widehat{\otimes}_L\mathcal{N}\widehat{\otimes}_L H \xrightarrow{\sigma_{23}} \mathcal{M}\widehat{\otimes}_L\mathcal{N}\widehat{\otimes}_L H\widehat{\otimes}_L H \xrightarrow{\mathrm{id}\widehat{\otimes}m} \mathcal{M}\widehat{\otimes}_L\mathcal{N}\widehat{\otimes}_L H$$

where $m : H\widehat{\otimes}_L H \to H$ denotes the multiplication map.

2.9 Quasi-abelian categories

We finish this chapter by recalling some of Schneiders' theory of quasi-abelian categories. For a more detailed treatment, see [77] and also [16, Section 4]. This theory provides a general framework for the issues to do with strictness that we saw already. It also explains how to do homological algebra on categories such as the category of Banach spaces.

Notation. From now on, $\mathbf{Ban}_L$ denotes the category of Banach spaces, with morphisms given by all continuous linear maps.

Definition 2.9.1. Let $\mathcal{C}$ be an additive category with kernels and cokernels.

(i) We say that a morphism $f : E \to F$ in $\mathcal{C}$ is *strict* if the induced morphism

$$\mathrm{Coim}(f) \to \mathrm{Im}(f)$$

is an isomorphism, where $\mathrm{Im}(f)$ is the kernel of the morphism $F \to \mathrm{coker}(f)$ and $\mathrm{Coim}(f)$ is the cokernel of the morphism $\ker(f) \to E$.

(ii) We say that $\mathcal{C}$ is *quasi-abelian* if it satisfies the following:

- In a cartesian square
$$\begin{array}{ccc} E' & \xrightarrow{f'} & F' \\ \downarrow & & \downarrow \\ E & \xrightarrow{f} & F \end{array}$$
if f is a strict epimorphism, then so is f'.
- In a co-cartesian square
$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \downarrow & & \downarrow \\ E' & \xrightarrow{f'} & F' \end{array}$$
if f is a strict monomorphism, then so is f'.

Example 2.9.2. Of course, abelian categories trivially satisfy the above definition since all morphisms are then strict. Moreover, $\mathbf{Ban}_L$ and $\mathbf{Mod}(A)$ are quasi-abelian by [16, Appendix A, Lemma A.30] and [77, Prop 1.5.1] respectively. Furthermore, by definition a morphism in $\mathbf{Mod}(A)$ or $\mathbf{Comod}(C)$ is strict if and only if it is strict in $\mathbf{Ban}_L$.

Remark 2.9.3. Note that in $\mathbf{Ban}_L$ (and in $\mathbf{Mod}(A)$ and $\mathbf{Comod}(C)$), while the kernel of a morphism $f : X \to Y$ is just the usual algebraic kernel (which is automatically closed), the cokernel of the same map is the canonical projection $Y \to Y/\overline{\mathrm{Im}(f)}$. Thus the categorical image of f in $\mathbf{Ban}_L$ (i.e. the kernel of the cokernel) is in fact the closure of the set theoretical image $\mathrm{Im}(f)$.

We now quickly recall explicitly what (co)cartesian squares are in $\mathbf{Ban}_L$ (see [16, Appendix A, Lemma A.30] and its proof). A cartesian square is a commutative diagram

$$\begin{array}{ccc} E' & \xrightarrow{f'} & F' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ E & \xrightarrow{f} & F \end{array}$$

in $\mathbf{Ban}_L$ where $E' = \ker((f, -g) : E \oplus F' \to F)$ and the maps f' and g' are just the restriction of the projection maps $E \oplus F' \to E$ and $E \oplus F' \to F'$ respectively.

A cocartesian square is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ E' & \xrightarrow{f'} & F' \end{array}$$

in $\mathbf{Ban}_L$ where $F' = (E' \oplus F)/\overline{\mathrm{Im}((g, -f) : E \to E' \oplus F)}$, i.e. F' is the cokernel of $(g, -f) : E \to E' \oplus F$. The maps f' and g' are then just obtained by post-composing the canonical embeddings $E' \to E' \oplus F$ and $F \to E' \oplus F$ respectively with the quotient map $E' \oplus F \to F'$.

Since taking direct sums, kernels and cokernels are constructions that can be done completely identically in $\mathbf{Mod}(A)$ and $\mathbf{Comod}(C)$, we see that the (co)cartesian squares in these categories are described completely identically as above. We now have:

Lemma 2.9.4. *Let C be a Banach coalgebra. Then $\mathbf{Comod}(C)$ is quasi-abelian.*

Proof. This category is clearly additive and, moreover, it follows from Lemma 2.8.18 that it has kernels and cokernels. By the above comments, the forgetful functor $\mathbf{Comod}(C) \to \mathbf{Ban}_L$ sends (co)cartesian squares to (co)cartesian squares. Thus, since a morphism in $\mathbf{Comod}(C)$ is strict if and only if it is strict in $\mathbf{Ban}_L$, it follows that $\mathbf{Comod}(C)$ is quasi-abelian. $\square$

We now explain how to do homological algebra in quasi-abelian categories.

Definition 2.9.5. Let $\mathcal{C}$ and $\mathcal{E}$ be quasi-abelian categories.

(i) A null sequence $E \xrightarrow{e} E' \xrightarrow{e'} E''$ in $\mathcal{C}$ is called *exact* if the canonical map $\mathrm{Im}(e) \to \ker(e')$ is an isomorphism. If furthermore e (resp. e') is strict then we say that this complex is *strictly exact* (resp. *strictly coexact*). A complex $E_1 \to \cdots \to E_n$ is called exact, resp. strictly (co)exact if each subsequence $E_{i-1} \to E_i \to E_{i+1}$ is exact, resp. strictly (co)exact.

(ii) An additive functor $F : \mathcal{C} \to \mathcal{E}$ is called *left exact* if it sends every strictly exact sequence

$$0 \to E \to E' \to E'' \to 0$$

in $\mathcal{C}$ to a strictly exact sequence

$$0 \to F(E) \to F(E') \to F(E'')$$

in $\mathcal{E}$. In other words F is left exact if it preserves kernels of strict morphisms.

We say that F is *strongly left exact* if it sends every strictly exact sequence

$$0 \to E \to E' \to E''$$

in $\mathcal{C}$ to a strictly exact sequence

$$0 \to F(E) \to F(E') \to F(E'')$$

in $\mathcal{E}$. In other words F is strongly left exact if it preserves kernels of arbitrary morphisms.

(iii) Similarly, we say that F is *right exact* if it sends every strictly exact sequence

$$0 \to E \to E' \to E'' \to 0$$

in $\mathcal{C}$ to a strictly coexact sequence

$$F(E) \to F(E') \to F(E'') \to 0$$

in $\mathcal{E}$. There is an analogous notion of strongly right exact functors.

(iv) We say that F is *exact* if it is both left exact and right exact, i.e. it sends every strictly exact sequence

$$0 \to E \to E' \to E'' \to 0$$

in $\mathcal{C}$ to a strictly exact sequence

$$0 \to F(E) \to F(E') \to F(E'') \to 0$$

in $\mathcal{E}$.

(v) An object I in $\mathcal{C}$ is called *injective* if the functor $E \mapsto \mathrm{Hom}(E, I)$ is exact, i.e. for any strict monomorphism $E \to F$, the induced map $\mathrm{Hom}(F, I) \to \mathrm{Hom}(E, I)$ is surjective. Dually an object P is called *projective* if the functor $E \mapsto \mathrm{Hom}(P, E)$ is exact, i.e. for any strict epimorphism $E \to F$, the induced map $\mathrm{Hom}(P, E) \to \mathrm{Hom}(P, F)$ is surjective.

(vi) We say that $\mathcal{C}$ has *enough injectives* if for any object E in $\mathcal{C}$, there is a strict monomorphism $E \to I$ where I is injective. Dually we say that $\mathcal{C}$ has *enough projectives* if for every E there is a strict epimorphism $P \to E$ from a projective object P.

Example 2.9.6. In $\mathbf{Ban}_L$, a map $f : X \to Y$ induces a continuous injection

$$X/\ker(f) \to \mathrm{Im}(f) \hookrightarrow \overline{\mathrm{Im}(f)},$$

and f is strict if and only if this injection is in fact an isomorphism of Banach spaces. This is equivalent to $\mathrm{Im}(f)$ being closed by Corollary 2.7.12.

Now, a sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

in $\mathbf{Ban}_L$ is exact if and only if the following conditions are satisfied:

- f is injective;
- $\ker(g) = \overline{\mathrm{Im}(f)}$; and
- g has dense image, i.e. $\overline{\mathrm{Im}(g)} = Z$.

The sequence is strict exact when f and g are furthermore strict. By the above criterion for strictness, we see that these three conditions then become that f is injective, g is surjective and $\ker(g) = \mathrm{Im}(f)$, i.e. the sequence is exact as a sequence of L-vector spaces. Hence we see that a sequence

$$0 \to X \to Y \to Z \to 0$$

in $\mathbf{Ban}_L$ is strict exact if and only if it is exact as a sequence of L-vector spaces. We will simply say in such a setting that the sequence is algebraically exact. The same will hold in $\mathbf{Mod}(A)$ and $\mathbf{Comod}(C)$ for a Banach algebra A and a Banach coalgebra C respectively.

Lemma 2.9.7 ([16, Lemma 4.25 & Appendix A, Lemma A.42])**.** *The quasi-abelian categories $\mathbf{Ban}_L$ and $\mathbf{Mod}(A)$ have enough injectives.*

We want to show that the same result holds in the category $\mathbf{Comod}(C)$. First we need a general result. The following is a well-known fact:

Lemma 2.9.8 ([11, Lemma 12.26.1])**.** *If $F : \mathcal{C} \to \mathcal{D}$ is an additive functor between abelian categories which preserves monomorphisms, and if it has a right adjoint $G : \mathcal{D} \to \mathcal{C}$, then G sends injective objects to injective objects*

This result has an obvious generalisation to quasi-abelian categories given as follows:

Lemma 2.9.9 ([16, Lemma 4.26])**.** *If $F : \mathcal{C} \to \mathcal{D}$ is an additive functor between quasi-abelian categories which preserves strict monomorphisms, and if it has a right adjoint $G : \mathcal{D} \to \mathcal{C}$, then G sends injective objects to injective objects.*

Suppose now that C is a Banach coalgebra. Then there is an adjunction (ϕ^*, ϕ_*) between $\mathbf{Comod}(C)$ and $\mathbf{Ban}_L$. Namely ϕ^* is the forgetful functor while $\phi_* : \mathcal{M} \mapsto \mathcal{M}\widehat{\otimes}_L C$ with coaction $\mathrm{id}_{\mathcal{M}} \widehat{\otimes}\Delta$. The bijection giving rise to this sends a map $f : \mathcal{M} \to \mathcal{N}\widehat{\otimes}_L C$ in $\mathbf{Comod}(C)$ to $(\mathrm{id}_{\mathcal{N}} \widehat{\otimes}\varepsilon) \circ f : \mathcal{M} \to \mathcal{N}$. The inverse sends a map $g : \mathcal{M} \to \mathcal{N}$ in $\mathbf{Ban}_L$ to $(g\widehat{\otimes}\,\mathrm{id}_C) \circ \rho_{\mathcal{M}} : \mathcal{M} \to \mathcal{N}\widehat{\otimes}_L C$ in $\mathbf{Comod}(C)$. That these are inverse to each other is just a formal consequence of the comodule axiom $(\mathrm{id}_{\mathcal{M}} \widehat{\otimes}\varepsilon) \circ \rho_{\mathcal{M}} = \mathrm{id}_{\mathcal{M}}$. Having established this we have:

Proposition 2.9.10. *The category $\mathbf{Comod}(C)$ has enough injectives.*

Proof. Note that since the forgetful functor ϕ^* preserves strict monomorphisms, and since ϕ_* is its right adjoint, it follows from Lemma 2.9.9 that ϕ_* preserves injective objects. Now let $\mathcal{M}$ be an object of $\mathbf{Comod}(C)$. Since $\mathbf{Ban}_L$ has enough injectives there exists a strict monomorphism $f : \mathcal{M} \hookrightarrow I$ of Banach spaces where I is injective. This induces a map $\hat{f} := f\widehat{\otimes}\,\mathrm{id}_C : \mathcal{M}\widehat{\otimes}_L C \to I\widehat{\otimes}_L C$ where we view $\mathcal{M}\widehat{\otimes}_L C$ as $\phi_*(\phi^*(\mathcal{M}))$. By the above

$\phi_*(I) = I\widehat{\otimes}_L C$ is injective in $\mathbf{Comod}(C)$, and we have the adjunction map $\rho : \mathcal{M} \to \mathcal{M}\widehat{\otimes}_L C$ which is just the comodule map. Therefore we have a morphism

$$\iota := \hat{f} \circ \rho : \mathcal{M} \to I\widehat{\otimes}_L C$$

in $\mathbf{Comod}(C)$ from $\mathcal{M}$ to an injective object. We claim that it is a strict monomorphism in $\mathbf{Ban}_L$, which implies the result. Now note that since ρ has a left inverse given by $1\widehat{\otimes}\varepsilon$, it follows by Lemma 2.7.20(iii) that ρ is a strict monomorphism. Moreover $\hat{f}$ is a strict monomorphism by Corollary 2.8.10. It now follows from Lemma 2.7.20(ii) that ι is a strict monomorphism as well. $\square$

Having defined what it means to have enough injectives in quasi-abelian categories, we now turn to derived categories and derived functors.

Definition 2.9.11. Let $\mathcal{C}$ be a quasi-abelian category.

(i) Let $K(\mathcal{C})$ be the homotopy category of $\mathcal{C}$, i.e. the category of complexes modulo homotopies. Then the *derived category* of $\mathcal{C}$ is defined to be

$$D(\mathcal{C}) = K(\mathcal{C})/N(\mathcal{C})$$

where $N(\mathcal{C})$ is the full subactegory of strictly exact sequences.

(ii) Let $F : \mathcal{C} \to \mathcal{E}$ be an additive functor between $\mathcal{C}$ and another quasi-abelian category $\mathcal{E}$. A full additive subcategory $\mathcal{I}$ of $\mathcal{C}$ is called *F-injective* if:

(1) for any object V of $\mathcal{C}$ there is a strict monomorphism $V \to I$ where I is an object of $\mathcal{I}$

(2) for any strictly exact sequence

$$0 \to V' \to V \to V'' \to 0$$

in $\mathcal{C}$, if V and V'' are objects of $\mathcal{I}$, then V' is as well

(3) for any strictly exact sequence

$$0 \to V' \to V \to V'' \to 0$$

in $\mathcal{C}$ where V, V' and V'' are objects of $\mathcal{I}$, the sequence

$$0 \to F(V') \to F(V) \to F(V'') \to 0$$

is strictly exact in $\mathcal{E}$

(iii) Given $F : \mathcal{C} \to \mathcal{E}$ as above, we say that F is *right derivable* if it has a right derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{E})$ which satisfies the usual universal property.

(iv) Assume F is right derivable. We say that an object I of $\mathcal{C}$ is *F-acyclic* if $RF(I) \cong F(I)$.

We now introduce the left heart of the derived category. We leave the definitions to the literature since we will not need them. By [77, Definition 1.2.17], there is a left t-structure on the derived category $D(\mathcal{C})$ of a quasi-abelian category $\mathcal{C}$, whose heart we denote by $\mathcal{LH}(\mathcal{C})$, called the *left heart* of the quasi-abelian category $\mathcal{C}$. The left heart is an abelian category and there is a natural functor $I : \mathcal{C} \to \mathcal{LH}(\mathcal{C})$. This satisfies nice properties:

Lemma 2.9.12. [2]

(i) ([77, Corollary 1.2.27]) *The functor I is a fully faithful embedding. Moreover a sequence in $\mathcal{C}$ is strictly exact if and only if it's exact in $\mathcal{LH}(\mathcal{C})$.*

(ii) ([77, Proposition 1.2.31]) *The functor I induces an equivalence of derived categories $D(I) : D(\mathcal{C}) \to D(\mathcal{LH}(\mathcal{C}))$.*

(iii) ([77, Corollary 1.2.19]) *For $n \in \mathbb{Z}$ let $LH^n : D(\mathcal{C}) \to \mathcal{LH}(\mathcal{C})$ denote the n-th cohomology functor. Then, for $E \in D(\mathcal{C})$, we have that $LH^n(E) = 0$ if and only if E is strictly exact in degree n.*

We gather all the important results that we need in the following:

Proposition 2.9.13. *Let $F : \mathcal{C} \to \mathcal{E}$ be an additive functor between quasi-abelian categories $\mathcal{C}$ and $\mathcal{E}$.*

(i) ([77, Prop 1.3.5]) *Assume that $\mathcal{C}$ has an F-injective subcategory. Then F has a right derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{E})$ (in this situation we say that F is* explicitly right derivable*).*

(ii) ([77, Remark 1.3.21]) *Assume that $\mathcal{C}$ has enough injectives. Then the full subcategory of injective objects is an F-injective subcategory.*

(iii) ([77, Remark 1.3.7]) *Assume that F has a right derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{E})$, and suppose that for any object V of $\mathcal{C}$, there is a monomorphism $V \to I$ where I is F-acyclic. Then the F-acyclic objects form an F-injective subcategory.*

(iv) ([77, Proposition 1.3.8 & Proposition 1.3.14]) *Assume that F is explicitly right derivable and let $\mathcal{I}$ be an F-injective subcategory of $\mathcal{C}$. Suppose that F is strongly left exact and that, for any monomorphism $I_0 \to I_1$ in $\mathcal{C}$ between objects of $\mathcal{I}$, $FI_0 \to FI_1$ is a monomorphism. Then F extends to an explicitly right derivable left exact functor $G : \mathcal{LH}(\mathcal{C}) \to \mathcal{LH}(\mathcal{E})$ such that $RG \cong RF$.*

[2] the reader may for our purposes take the functor I and the properties in the Lemma as a definition of the left heart

Chapter 3

Fréchet–Stein algebras and rigid analytic quantum groups

In this chapter we introduce the Fréchet algebras $\widehat{U_q}$, $\widehat{\mathcal{O}_q}$ and prove Theorem A from the Introduction.

Convention. In this chapter, the term “flat” will be used to mean “flat on both sides” unless explicitly stated otherwise. All of our filtrations on modules or algebras will be positive and exhaustive unless specified otherwise. Furthermore, given a ring S, a subring F_0S such that S is generated over F_0S by some elements $x_1, \dots, x_n$ which normalise F_0S, and integers $d_1, \dots, d_n \geq 1$, there is a ring filtration on S by F_0S-submodules given by setting

$$F_tS = F_0S \cdot \{x_{i_1} \cdots x_{i_r} : \sum_{j=1}^{r} d_{i_j} \leq t\}$$

for each $t \geq 0$. In such a setting, we will simply say ‘the filtration given by assigning each x_i degree d_i’ to refer to this filtration.

3.1 The functor $M \mapsto \widehat{M_L}$

We begin by recalling constructions from [8, Section 6.7], which were written in terms of R-algebras but extend identically to R-modules. Recall the notation from Definition 2.6.5. If M is a torsion-free, filtered R-module, let $\widehat{M_{n,L}} := \widehat{M_n} \otimes_R L$ for each $n \geq 0$. This is an L-Banach space by Lemma 2.7.14, with unit ball $\widehat{M_n}$. To simplify notation, we write $\widehat{M_L}$ for $\widehat{M_{0,L}}$.

Now, we have a descending chain

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots$$

which induces an inverse system of L-Banach spaces and continuous linear maps

$$\widehat{M_L} = \widehat{M_{0,L}} \leftarrow \widehat{M_{1,L}} \leftarrow \widehat{M_{2,L}} \leftarrow \cdots$$

whose inverse limit we write as

$$\widehat{M_L} := \varprojlim \widehat{M_{n,L}}.$$

The maps $\widehat{M_L} \to \widehat{M_{n,L}}$ induce continuous semi-norms $||\cdot||_n$ on $\widehat{M_L}$, such that the completion of $\widehat{M_L}$ with respect to $||\cdot||_n$ is $\widehat{M_{n,L}}$. Hence $\widehat{M_L}$ is an L-Fréchet space. Thus we have defined a functor $M \mapsto \widehat{M_L}$ from torsion-free filtered R-modules to the category of L-Fréchet spaces.

We now apply the above construction to certain lattices in the quantum groups we've defined in the previous chapter. Let U be the De Concini-Kac integral form of the quantum group. Recall it was the R-subalgebra of U_q generated by the E_{α_i}'s, F_{α_j}'s and the K's. We filter this algebra by setting $F_0U = R[K_\lambda : \lambda \in P]$ and giving each E_α and F_α degree 1. Then each deformation U_n is the R-subalgebra of U_q generated by the $\pi^n E_{\alpha_i}$'s, $\pi^n F_{\alpha_j}$'s and the K's.

Note that by the definition of the Hopf algebra structure on U_q given in Proposition 2.4.2, we see that each U_n is an R-Hopf subalgebra of U_q. Indeed, by using Lemma 2.6.8, this follows from the fact that the Hopf algebra maps on U are filtered maps, where for Δ we give $U \otimes_R U$ the tensor filtration.

Definition 3.1.1. We let $\widehat{U_{q,n}} := \widehat{U_{n,L}}$ and $\widehat{U_q} := \widehat{U_L} = \varprojlim \widehat{U_{q,n}}$ where we give U the above filtration.

Now recall the integral form $\mathcal{A}_q$ of the quantum group $\mathcal{O}_q$ from Definition 2.5.7. We saw in Proposition 2.5.17 that $\mathcal{A}_q$ is generated by the matrix coefficients of certain U^{res}-lattices inside the fundamental representations of U_q. So we see that by choosing weight bases for these lattices, the generators $x_1, \dots, x_r$ of $\mathcal{O}_q$ from equation (2.5) (see the discussion following Proposition 2.5.3) lie in $\mathcal{A}_q$ and generate it as an R-algebra. We now give the filtration to $\mathcal{A}_q$ given by assigning to each x_i degree 1. So the n-th deformation is the R-subalgebra generated by all the $\pi^n x_i$.

Definition 3.1.2. We let $\widehat{\mathcal{O}_q} := \widehat{(\mathcal{A}_q)_L}$ where we give $\mathcal{A}_q$ the above filtration.

We will now show that $\widehat{U_q}$ and $\widehat{\mathcal{O}_q}$ are Hopf algebras in a suitable sense, when working in the category of L-Fréchet spaces. We will need the following elementary result:

Lemma 3.1.3. *Let M, N be two R-modules. Then we have canonical isomorphisms*

$$(M/\pi^a M) \otimes_R (N/\pi^a N) \cong (M/\pi^a M) \otimes_R N \cong M \otimes_R (N/\pi^a N) \cong (M \otimes_R N)/\pi^a (M \otimes_R N)$$

for any $a \geq 1$.

Proof. By tensoring the short exact sequence

$$0 \to \pi^a M \to M \to M/\pi^a M \to 0$$

with N, we obtain an exact sequence

$$\pi^a M \otimes_R N \to M \otimes_R N \to M/\pi^a M \otimes_R N \to 0.$$

Thus, since the image of $\pi^a M \otimes_R N$ in $M \otimes_R N$ equals $\pi^a(M \otimes_R N)$, we see that

$$(M/\pi^a M) \otimes_R N \cong (M \otimes_R N)/\pi^a(M \otimes_R N).$$

Similarly $M \otimes_R (N/\pi^a N) \cong (M \otimes_R N)/\pi^a(M \otimes_R N)$ by interchanging M and N. Finally, if we tensor the short exact sequence

$$0 \to \pi^a N \to N \to N/\pi^a N \to 0$$

with $M/\pi^a M$, we obtain an exact sequence

$$(M/\pi^a M) \otimes_R \pi^a N \to (M/\pi^a M) \otimes_R N \to (M/\pi^a M) \otimes_R (N/\pi^a N) \to 0$$

where the left hand side map clearly has image 0. Thus we get the required isomorphism. □

Proposition 3.1.4. *Let M and N be torsion-free R-modules. Then there is a canonical isomorphism of L-Banach spaces*

$$\widehat{M_L} \widehat{\otimes}_L \widehat{N_L} \cong \widehat{(M \otimes_R N)_L}.$$

Moreover when M and N are R-algebras, this map is an algebra isomorphism. In particular, $M \mapsto \widehat{M_L}$ is a monoidal functor between the category of torsion-free R-modules and the category of L-Banach spaces.

Proof. Note that $\widehat{M_L} \otimes_L \widehat{N_L} \cong (\widehat{M} \otimes_R \widehat{N}) \otimes_R L$ and, by the Lemma, we have natural isomorphisms

$$\begin{aligned}(\widehat{M} \otimes_R \widehat{N})/\pi^a(\widehat{M} \otimes_R \widehat{N}) &\cong \widehat{M}/\pi^a\widehat{M} \otimes_R \widehat{N}/\pi^a\widehat{N} \\ &\cong M/\pi^a M \otimes_R N/\pi^a N \\ &\cong (M \otimes_R N)/\pi^a(M \otimes_R N)\end{aligned}$$

for all $a \geq 1$. Thus we see that $\widehat{M \otimes_R N}$ is canonically isomorphic to the π-adic completion of $\widehat{M} \otimes_R \widehat{N}$. Hence we see that $\widehat{(M \otimes_R N)_L}$ is the completion of $\widehat{M_L} \otimes_L \widehat{N_L}$ with respect to the gauge norm associated to $\widehat{M} \otimes_R \widehat{N}$ by Proposition 2.7.8. By Proposition 2.8.9(ii), the latter topology is the same as the tensor product topology on $\widehat{M_L} \otimes_L \widehat{N_L}$, and so we get the result.

In the case where $M = A$ and $N = B$ are algebras, it is clear from the above that the isomorphism preserves the algebra structure. □

Corollary 3.1.5. *Suppose that H is a torsion-free R-Hopf algebra. Then $\widehat{H_L} := \widehat{H} \otimes_R L$ is a Banach Hopf algebra.*

Proof. By the Proposition, after applying the π-adic completion functor to the Hopf algebra maps Δ, ε and S, and extending scalars to L, we obtain continuous linear maps

$$\widehat{\Delta} : \widehat{H_L} \to \widehat{H_L} \widehat{\otimes}_L \widehat{H_L},\ \widehat{\varepsilon} : \widehat{H_L} \to L \text{ and } \widehat{S} : \widehat{H_L} \to \widehat{H_L}.$$

By definition, these map satisfy the Hopf algebra axioms since they do on the dense subset H_L. □

We introduce the following notation: write $\widehat{\mathcal{O}_q} := \widehat{(\mathcal{A}_q)_L}$. From the above we immediatly obtain:

Corollary 3.1.6. *The Banach algebras $\widehat{\mathcal{O}_q}$ and $\widehat{U_{q,n}}$ $(n \geq 0)$ are Banach Hopf algebras.*

Example 3.1.7. When $G = \mathrm{SL}_2$ i.e. when $\mathfrak{g} = \mathfrak{sl}_2$, we can give an explicit description of $\widehat{\mathcal{O}_q}$. In that case the only fundamental representation of U_q is two dimensional with basis v_1, v_2 such that

$$Ev_1 = 0 = Fv_2 \quad Ev_2 = v_1 \quad Fv_1 = v_2 \quad Kv_1 = q^{\frac{1}{2}}v_1 \quad Kv_2 = q^{\frac{-1}{2}}v_2.$$

Note that this is well-defined since, by our assumption that $p > 2$, it follows from Hensel's lemma that q has a square root in R. The matrix coefficients with respect to that basis are denoted by $x_{11}, x_{12}, x_{21}, x_{22}$ and they generate $\mathcal{O}_q$. As is customary we denote these generators by a, b, c and d respectively. The complete set of relations for $\mathcal{O}_q$ is given by

$$\begin{aligned} &ab = qba, \quad ac = qca, \quad bc = cb, \quad bd = qdb, \\ &cd = qdc, \quad ad - da = (q - q^{-1})bc, \quad ad - qbc = 1. \end{aligned}$$

(see [25, Theorem I.7.16]).

So in this case $\mathcal{A}_q$ is the R-algebra generated by a, b, c, d. In fact $\mathcal{A}_q$ is a free R-module and

$$\mathcal{S} = \{a^l b^m c^s, b^m c^s d^t : l, m, s \geq 0 \text{ and } t > 0\}$$

is an R-basis of $\mathcal{A}_q$ (see [30, Lemma 1.1] which is stated over fields but works identically with the integral form). Concretely, one can identify $\widehat{\mathcal{O}_q}$ as the ring

$$\widehat{\mathcal{O}_q} = \Big\{ \sum_{l,m,s \geq 0} \lambda_{lms} a^l b^m c^s + \sum_{\substack{p,t \geq 0 \\ r > 0}} \mu_{ptr} b^p c^t d^r : |\lambda_{lms}| \to 0 \text{ as } l + m + s \to \infty$$
$$\text{and } |\mu_{ptr}| \to 0 \text{ as } p + t + r \to \infty \Big\}.$$

This is an L-Banach algebra with norm

$$\Big|\Big| \sum \lambda_{lms} a^l b^m c^s + \sum \mu_{ptr} b^p c^t d^r \Big|\Big| := \sup_{l,m,s,p,t,r} \{\lambda_{lms}, \mu_{ptr}\}.$$

We will later give an explicit description of $\widehat{U_{q,n}}$ for n large enough.

We now move on to the Fréchet completions:

Theorem 3.1.8. *The functor $M \mapsto \widehat{M_L}$, from the category of torsion-free filtered R-modules to the category of Fréchet spaces, is monoidal.*

Proof. From Proposition 2.8.12 we see that for any two torsion-free filtered R-modules M

and N, there is a canonical isomorphism of L-Fréchet spaces

$$\widehat{M_L}\widehat{\otimes}_L\widehat{N_L} \cong \varprojlim \widehat{M_{n,L}}\widehat{\otimes}_L\widehat{N_{n,L}}$$

which is an algebra isomorphism when M and N are R-algebras. But Proposition 3.1.4 and Lemma 2.6.8 now imply that

$$\widehat{M_L}\widehat{\otimes}_L\widehat{N_L} \cong \widehat{(M \otimes_R N)_L}$$

as required. $\square$

Corollary 3.1.9. *The Fréchet algebra $\widehat{U_q}$ is a Fréchet Hopf algebra.*

Proof. This follows from the Theorem because U is a filtered Hopf algebra, meaning that Δ, ε and S are filtered maps (where for ε we give R the trivial filtration). Indeed, for each $n \geq 0$, we have maps

$$\Delta_n : U_n \to U_n \otimes_R U_n, \varepsilon_n : U_n \to R, \text{ and } S : U_n \to U_n$$

obtained by taking n-th deformation, i.e. by taking restriction. Then the completions of these maps make $\widehat{U_{q,n}}$ into a Banach Hopf algebra with maps $\widehat{\Delta_n}$, $\widehat{\varepsilon}$ and $\widehat{S_n}$, in such a way that these form inverse systems of maps $(\widehat{U_{q,n}})_{n\geq 0} \to (\widehat{U_{q,n}}\widehat{\otimes}_L\widehat{U_{q,n}})_{n\geq 0}$, $(\widehat{U_{q,n}})_{n\geq 0} \to L$ and $(\widehat{U_{q,n}})_{n\geq 0} \to (\widehat{U_{q,n}})_{n\geq 0}$ respectively. Passing to the inverse limit and applying the Theorem gives maps

$$\widehat{\Delta} : \widehat{U_q} \to \widehat{U_q}\widehat{\otimes}_L\widehat{U_q},\ \widehat{\varepsilon} : \widehat{U_q} \to L \text{ and } \widehat{S} : \widehat{U_q} \to \widehat{U_q}$$

which satisfy the Hopf algebra axioms since they do on the dense subset U_q. $\square$

We know that $\mathcal{A}_q$ is a Hopf algebra, however the corresponding Hopf algebra maps are not all filtered R-module homomorphisms on $\mathcal{A}_q$, so we can't immediately deduce from our previous methods that $\widehat{\mathcal{O}_q}$ has a Hopf algebra structure. On the other hand, we see from Lemma 2.5.2 that the counit restricted to $\mathcal{A}_q$ is a filtered R-map $\mathcal{A}_q \to R$ and so gives rise to a map $\widehat{\epsilon} : \widehat{\mathcal{O}_q} \to L$. For the antipode and comultiplication, we can "shift" the deformations to make things work.

Indeed, from Lemma 2.5.2 again, we have $\Delta(F_n\mathcal{A}_q) \subseteq F_n\mathcal{A}_q \otimes_R F_n\mathcal{A}_q$ for all $n \geq 0$. But then it follows that for all $n \geq 0$ we have

$$\Delta((\mathcal{A}_q)_{2n}) \subseteq (\mathcal{A}_q)_n \otimes_R (\mathcal{A}_q)_n.$$

Taking π-adic completions we see that Δ induces maps

$$\widehat{\Delta}_n : \widehat{(\mathcal{A}_q)_{2n,L}} \to \widehat{(\mathcal{A}_q)_{n,L}}\widehat{\otimes}_L\widehat{(\mathcal{A}_q)_{n,L}}$$

by Proposition 3.1.4. Taking inverse limits we obtain a map

$$\widehat{\Delta} : \widehat{\mathcal{O}_q} \to \widehat{\mathcal{O}_q}\widehat{\otimes}_L\widehat{\mathcal{O}_q}$$

by Theorem 3.1.8. We now move to the antipode. It's not necessarily clear that it's a filtered map on $\mathcal{A}_q$, so we let

$$d = \max_{1 \le i \le r} \{\min\{t : S(x_i) \in F_t \mathcal{A}_q\}\}.$$

It follows that $S((\mathcal{A}_q)_{nd}) \subseteq (\mathcal{A}_q)_n$ for all $n \ge 0$. Taking π-adic completions we see that S induces maps

$$\widehat{S}_n : \widehat{(\mathcal{A}_q)_{nd,L}} \to \widehat{(\mathcal{A}_q)_{n,L}}.$$

Taking inverse limits we obtain a map

$$\widehat{S} : \widehat{\mathcal{O}_q} \to \widehat{\mathcal{O}_q}.$$

We see that the maps $\widehat{\epsilon}$, $\widehat{S}$ and $\widehat{\Delta}$ make $\widehat{\mathcal{O}_q}$ into a Hopf algebra, as desired, since all the Hopf algebra relations are satisfied on the dense subspace $\mathcal{O}_q$.

Remark 3.1.10. Note that the above shifts really are to be expected. Indeed, for example in the case $G = \mathrm{SL}_n(L)$, the algebra we construct is meant to be a quantum analogue of the global sections of the structure sheaf of the analytification of G. If $\mathcal{O}$ denotes the coordinate algebra of $\mathrm{SL}_n(R)$, this ring of global sections is given by the inverse limit of the Banach algebras $\widehat{\mathcal{O}_{m,L}}$, which correspond to the functions on G which are analytic on $\mathrm{SL}_n(\pi^{-m}R)$. For $m > 0$, since that subset of G is not a subgroup, the algebra $\widehat{\mathcal{O}_{m,L}}$ is not a Hopf algebra. On the other hand matrix multiplication defines a map

$$\mathrm{SL}_n(\pi^{-m}R) \times \mathrm{SL}_n(\pi^{-m}R) \to \mathrm{SL}_n(\pi^{-2m}R)$$

which induces a map $\Delta : \widehat{\mathcal{O}_{2m,L}} \to \widehat{\mathcal{O}_{m,L}} \widehat{\otimes}_L \widehat{\mathcal{O}_{m,L}}$. Our quantum situation very much mirrors this.

3.2 Fréchet–Stein algebras

We start with a definition.

Definition 3.2.1. Following [75, Section 3] we say that an L-algebra $\mathcal{U}$ is *L-Fréchet-Stein* if there is a tower $\mathcal{U}_0 \leftarrow \mathcal{U}_1 \leftarrow \mathcal{U}_2 \leftarrow \cdots$ of Noetherian Banach algebras with dense images such that

(i) $\mathcal{U}_n$ is a flat $\mathcal{U}_{n+1}$-module for all $n \ge 0$; and

(ii) $\mathcal{U} \cong \varprojlim \mathcal{U}_n$.

Our main aim is to prove that the algebras $\widehat{\mathcal{O}_q}$ and $\widehat{U_q}$ are Fréchet-Stein. The main difficulty in proving that an algebra satisfies the above definition is to show that the flatness condition in (i) holds. To do this we rely on two known results. The first one, due to Emerton, is the following:

Proposition 3.2.2 ([36, Proposition 5.3.10])**.** *Suppose that A is a left Noetherian R-algebra, π-adically separated, π-torsion free, and suppose that B is an R-subalgebra of*

A_L which contains A. Suppose B is equipped with an exhaustive R-algebra filtration $(F_\cdot)$ satisfying $F_0B = A$ and such that $\operatorname{gr} B$ is finitely generated as an A-algebra by central elements. Then $\widehat{A_L}$ and $\widehat{B_L}$ are left Noetherian and $\widehat{B_L}$ is right flat over $\widehat{A_L}$.

Recall the notion of a deformable R-algebra from Definition 2.6.5. Our second method is due to Ardakov and Wadsley, and is using a certain class of deformable algebras as well the functor $M \mapsto \widehat{M_L}$ we defined earlier.

Theorem 3.2.3 ([8, Theorem 6.7])**.** *Let U be a deformable R-algebra such that $\operatorname{gr} U$ is commutative and Noetherian. Then $\widehat{U_L}$ is a Fréchet-Stein algebra.*

The issue with these methods is that the statements both involve some commutativity or centralness conditions that will not hold in the quantum setting. Therefore, in this section, we will prove certain non-commutative, or quantum, versions of these results.

We first generalise Theorem 3.2.3. The proofs from [8, Section 6.5 & 6.6] go through with only minor changes. Throughout, we will make the following assumptions:

(i) U is a deformable R-algebra such that $\operatorname{gr}_0 U$ and $\operatorname{gr} U$ are Noetherian;

(ii) there are elements $x_1, \ldots, x_r \in U$ such that

$$F_iU = F_0U \cdot \{x_1^{\alpha_1} \cdots x_r^{\alpha_r} : \sum_{j=1}^{r} \alpha_j d_j \leq i\}$$

for each $i \geq 0$, where $d_j = \deg x_j$, so that then $\operatorname{gr} U$ is finitely generated over $\operatorname{gr}_0 U$ by the symbols of $x_1, \ldots, x_r \in U$; and

(iii) the sequence $\overline{\pi^{d_1} x_1}, \ldots, \overline{\pi^{d_r} x_r}$, where $\overline{\pi^{d_i} x_i}$ denotes the image of $\pi^{d_i} x_i$ in $U_1/\pi U_1$, is polynormal.

Note that (i)-(iii) hold when U is a deformable R-algebra such that $\operatorname{gr} U$ is commutative and Noetherian by the proofs in [8, Sections 6.5 & 6.6].

Lemma 3.2.4. *If U satisfies (i) and (ii) as above, then so does U_n for all $n \geq 1$.*

Proof. This is a straightforward application of Lemma 2.6.7(i): (i) follows immediately because $\operatorname{gr} U_n \cong \operatorname{gr} U$ and $\operatorname{gr}_0 U_n = \operatorname{gr}_0 U$, while (ii) follows because

$$F_iU_n = F_0U \cdot \{(\pi^{nd_1} x_1)^{\alpha_1} \cdots (\pi^{nd_r} x_r)^{\alpha_r} : \sum_{j=1}^{r} \alpha_j d_j \leq i\}$$

from which we see that $\operatorname{gr} U_n$ is generated by the symbols of $\pi^{nd_1} x_1, \ldots, \pi^{nd_r} x_r$ over $\operatorname{gr}_0 U_n$. □

Proposition 3.2.5. *Let U be a deformable R-algebra satisfying condition (ii) above, and consider the ideal $I := U_1 \cap \pi U$.*

(a) The subspace filtration on U_1 of the π-adic filtration on U and the I-adic filtration on U_1 are topologically equivalent; and

(b) I is generated by π and $\pi^{d_j}x_j$ for $1 \leq j \leq n$.

Proof. It is clear from the definition of I that

$$\pi \in I \quad \text{and} \quad \pi^{d_j}x_j \in I \quad \text{for all} \quad 1 \leq j \leq n.$$

Let $d_0 := 1$. It follows from condition (ii) that $\pi^i F_i U$ is generated as an F_0U-module by monomials of the form

$$(\pi^{d_0})^{\alpha_0}(\pi^{d_1}x_1)^{\alpha_1} \cdots (\pi^{d_n}x_n)^{\alpha_n} \tag{3.1}$$

where $\alpha_j \geq 0$ for all $j = 0, \ldots, n$ and $\sum_{j=0}^{n} \alpha_j d_j = i$. For any integer $t \geq 0$ and $i \geq t \max d_j$, we have $(\sum_{j=0}^{n} \alpha_j) \max d_j \geq \sum_{j=0}^{n} \alpha_j d_j = i \geq t \max d_j$, so

$$(\pi^{d_0})^{\alpha_0}(\pi^{d_1}x_1)^{\alpha_1} \cdots (\pi^{d_n}x_n)^{\alpha_n} \in I^t$$

since $\pi \in I$ and $\pi^{d_j}x_j \in I$ for all $1 \leq j \leq m$. Hence by Lemma 2.6.7(ii) we have

$$U_1 \cap \pi^{t \max d_j} U = \sum_{i \geq t \max d_j} \pi^i F_i U \subseteq I^t \subseteq U_1 \cap \pi^t U$$

since I is an F_0U-submodule of U, thus proving (a).

For (b), by Lemma 2.6.7(ii) we have $I = \sum_{i \geq 1} \pi^i F_i U$. But we know from (3.1) above that, for $i \geq 1$, $\pi^i F_i U$ is generated as an F_0U-module by elements which are in the ideal generated by π and $\pi^{d_j}x_j$ for $1 \leq j \leq n$. The result follows. □

We can now prove our version of [8, Theorem 6.6]. Their proof goes through unchanged except for our use of condition (iii) which replaces their commutativity constraint.

Theorem 3.2.6. *Let U be a deformable R-algebra satisfying conditions (i)-(iii). Then $\widehat{U_L}$ is flat over $\widehat{U_{1,L}}$.*

Proof. Since $\widehat{U_{1,L}} = \widehat{U_1} \otimes_R L$, it is enough to show that $\widehat{U_L}$ is flat as a module over $\widehat{U_1}$. By Proposition 3.2.5(i), the I-adic completion B of U_1 is isomorphic to the closure of the image of U_1 in $\widehat{U}$. Hence we have natural maps $\widehat{U_1} \to B \to \widehat{U_L}$. Observe that B is π-adically complete by the proof of [84, Theorem VIII.5.14], noting that ideals in B are I-adically closed by [55, Theorem II.2.1.2, Proposition II.2.2.1].

We observe that $B/\pi B$ is the $I/\pi U_1$-adic completion of $U_1/\pi U_1$. From Proposition 3.2.5(ii), the ideal $I/\pi U_1$ is generated by $\overline{\pi^{d_j}x_j}$ for $1 \leq j \leq n$. Hence it follows from condition (iii) and [64, Proposition D.V.1 & Remark D.V.2] that $I/\pi U_1$ has the Artin-Rees property. Thus we have that $B/\pi B$ is flat over $U_1/\pi U_1$ by [64, Property V.8)iii), page 301].

We now filter both $\widehat{U_1}$ and B π-adically. Since U_1 is π-torsion free, we have $\operatorname{gr} \widehat{U_1} \cong (U_1/\pi U_1)[t]$. In a similar way, since B is isomorphic to a subring of $\widehat{U}$ and so has no π-torsion, we have $\operatorname{gr} B \cong (B/\pi B)[t]$. Hence $\operatorname{gr} B$ is flat over $\operatorname{gr} \widehat{U_1}$. But this implies that B is a flat $\widehat{U_1}$-module by [75, Proposition 1.2], since both $\widehat{U_1}$ and B are π-adically complete.

We now consider the subspace filtration on U_1 induced from the π-adic filtration on U. We have $\operatorname{gr} U \cong \overline{U}[t]$ where $t = \operatorname{gr} \pi$ and $\overline{U} = U/\pi U$ has degree zero. Lemma 2.6.7(ii) implies that the image of $\operatorname{gr} U_1$ inside $\operatorname{gr} U$ is $\oplus_{j\geq 0} t^j \overline{F_j U}$ where $\overline{F_j U}$ denotes the image of $F_j U$ in $\overline{U}$. Note that $\operatorname{gr} U_1$ is Noetherian by [24, Corollary 1.3] and conditions (i) and (iii) since it is generated over $\overline{\operatorname{gr}_0 U}$ by the $t^{d_i}\overline{x_i}$. Now, as the quotient filtration $\overline{F_j U}$ on $\overline{U}$ is exhaustive, the localisation of this image obtained by inverting t is $\overline{U}[t,t^{-1}]$. But B is the completion of U_1 so

$$(\operatorname{gr} B)_t = (\operatorname{gr} U_1)_t = \overline{U}[t,t^{-1}] = \operatorname{gr} \widehat{U_L},$$

where $\widehat{U_L}$ is given the extension of the π-adic filtration $F_i\widehat{U_L} = \pi^{-i}\widehat{U}$ for every $i \in \mathbb{Z}$. Hence $\operatorname{gr}\widehat{U_L}$ is flat over $\operatorname{gr} B$. We can then invoke [75, Proposition 1.2] again to conclude that $\widehat{U_L}$ is flat over B. □

From this we deduce the main result we need.

Theorem 3.2.7. *Let U be a deformable R-algebra satisfying assumptions (i)-(ii), such that U_n satisfies (iii) for all $n \geq 0$. Then $\widehat{U_L}$ is a Fréchet-Stein algebra.*

Proof. By Lemma 3.2.4 each U_n satisfies conditions (i)-(iii). Now since $(U_n)_1 = U_{n+1}$ by Lemma 2.6.7(iii), we have by Theorem 3.2.6 that $\widehat{U_{n,L}}$ is a flat $\widehat{U_{n+1,L}}$-module. Moreover, each $\widehat{U_{n,L}}$ is Noetherian because $\operatorname{gr} U$ is Noetherian. Finally the image of each map $\widehat{U_{n+1,L}} \to \widehat{U_{n,L}}$ is dense for every $n \geq 0$ since it contains U_L. □

We now turn to the important notion of a coadmissible module:

Definition 3.2.8 ([75, Section 3])**.** Let $\mathcal{U} = \varprojlim \mathcal{U}_n$ be a Fréchet-Stein algebra. Then a $\mathcal{U}$-module $\mathcal{M}$ is called *coadmissible* if $\mathcal{M} \cong \varprojlim \mathcal{M}_n$ where, for each $n \geq 0$, $\mathcal{M}_n$ is a finitely generated $\mathcal{U}_n$-module and $\mathcal{U}_n \otimes_{\mathcal{U}_{n+1}} \mathcal{M}_{n+1} \cong \mathcal{M}_n$. The full subcategory of coadmissible modules is denoted by $\mathcal{C}(\mathcal{U})$.

Note that if $\mathcal{M}$ is a coadmissible module, then by Lemma 2.8.6 each $\mathcal{M}_n$ is a Banach $\mathcal{U}_n$-module, and so $\mathcal{M}$ naturally has the structure of a Fréchet space.

We summarise below the facts we'll need:

Proposition 3.2.9 ([75, Lemma 3.6 & Corollaries 3.1, 3.4 & 3.5])**.** *Let $\mathcal{U}$ be a Fréchet-Stein algebra and let $\mathcal{M}$ be a coadmissible $\mathcal{U}$-module.*

(i) For each $n \geq 0$, $\mathcal{M}_n \cong \mathcal{U}_n \otimes_{\mathcal{U}} \mathcal{M}$.

(ii) The category $\mathcal{C}(\mathcal{U})$ is an abelian subcategory of the category of all $\mathcal{U}$-modules; it is closed under direct sums and contains the finitely presented $\mathcal{U}$-modules.

(iii) Let $\mathcal{N}$ be a submodule of $\mathcal{M}$. Then the following are equivalent:

(1) $\mathcal{N}$ is coadmissible;

(2) $\mathcal{M}/\mathcal{N}$ is coadmissible; and

(3) $\mathcal{N}$ is closed in the above Fréchet topology.

(iv) A sum of two coadmissible submodules of $\mathcal{M}$ is coadmissible.

(v) Any finitely generated submodule of $\mathcal{M}$ is coadmissible.

(vi) Any module map between two coadmmissible module is strict with closed image.

The proof of the next result is essentially the proof of the first part of [75, Theorem 4.11] (see also [72, Theorem 4.3.3]).

Corollary 3.2.10. *Let U be a deformable R-algebra satisfying assumptions (i)-(ii), such that U_n satisfies (iii) for all $n \geq 0$. Then the natural map $U_L \to \widehat{U_L}$ is flat.*

Proof. We show right flatness, the proof of left flatness being completely analogous. Since π is central, for every $n \geq 0$ the ideal πU_n in U_n satisfies the Artin-Rees property and thus $\widehat{U_n}$ is flat over U_n by [64, Proposition D.V.1 & Property V.8)iii), page 301]. Hence it follows that $U_L \to \widehat{U_{n,L}}$ is flat for every $n \geq 0$. Next, by Theorem 3.2.7 we know that $\widehat{U_L}$ is Fréchet-Stein. It will suffice to show that for a left ideal $I \subset U_L$, the map $\widehat{U_L} \otimes_{U_L} I \to \widehat{U_L}$ is injective. But now, I is finitely generated and in fact finitely presented since U_L is Noetherian by Lemma 2.6.1. Thus $\widehat{U_L} \otimes_{U_L} I$ is finitely presented as well, and so coadmissible by Proposition 3.2.9(ii). Thus we have isomorphisms

$$\widehat{U_L} \otimes_{U_L} I \cong \varprojlim \left(\widehat{U_{n,L}} \otimes_{\widehat{U_L}} (\widehat{U_L} \otimes_{U_L} I) \right) \cong \varprojlim (\widehat{U_{n,L}} \otimes_{U_L} I)$$

Proposition 3.2.9(i). Now as $\widehat{U_{n,L}}$ is flat over U_L, it follows that $\widehat{U_{n,L}} \otimes_{U_L} I \to \widehat{U_{n,L}}$ is injective for every $n \geq 0$. The result then follows since projective limits preserve injections. $\square$

When it is not known whether the algebras we have at hand are deformable, we instead rely on techniques inspired from Emerton's result to prove that their completions are Fréchet-Stein. Again, the arguments from [36, 5.3.5-5.3.10] follow through with only minor changes. They mainly rely on some general lemmas that we do not write out here but reference throughout the proof.

Proposition 3.2.11. *Suppose that A is a left Noetherian R-algebra, π-adically separated, π-torsion free, and suppose that B is an R-subalgebra of A_L which contains A. Suppose B is equipped with an exhaustive R-algebra filtration $(F_\cdot)$ satisfying $F_0 B = A$ and such that $gr^F B$ is a q-commutative A-algebra. Then $\widehat{A_L}$ and $\widehat{B_L}$ are left Noetherian and $\widehat{B_L}$ is right flat over $\widehat{A_L}$.*

Proof. Note that $\widehat{A}$ is left Noetherian because A is left Noetherian, hence so is $\widehat{A_L}$. Furthermore, $\operatorname{gr} B$ is left Noetherian by Lemma 2.5.6. Now, following [36], for any left A-submodule M of A_L, we let $\iota_M : \widehat{A} \otimes_A M \to \widehat{A_L}$ be the natural map induced from the multiplication in $\widehat{A_L}$, and we let C denote the image of ι_B. It is shown in [36, Corollary 5.3.6] that C is an R-subalgebra of $\widehat{A_L}$. Let $G_i C$ denote the image of $\iota_{F_i B}$. It is shown in [36, Lemma 5.3.5] that $G_i C$ is equal to $F_i B + \widehat{A}$ and $C = B + \widehat{A}$, and so we see that $(G'_\cdot)$ is an exhaustive algebra filtration on C such that $G'_0 C = \widehat{A}$. Now, $F_{i-1} B = A_L \cap G_{i-1} C$ for all $i \geq 1$ (see [36, Lemma 5.3.5]) and so it follows that $F_{i-1} B = F_i B \cap (F_{i-1} B + \widehat{A})$. Hence the natural map $\operatorname{gr}_i^F B \to \operatorname{gr}_i^G C$ induced by $\iota_{F_i B}$ is an isomorphism. Thus we deduce from our assumptions that the associated graded ring $\operatorname{gr}^{G'} C$ is a q-commutative

$\widehat{A}$-algebra. Therefore by Lemma 2.5.6 we have that $\mathrm{gr}^{G'} C$ is left Noetherian, hence so is C.

The fact that $\widehat{B_L}$ is right flat over $\widehat{A_L}$ now follows easily. Indeed, since $C = B + \widehat{A}$ we see that $C_L = \widehat{A_L}$. Moreover $\widehat{B_L} \cong \widehat{C_L}$ (see [36, Lemma 5.3.8]). But the ideal generated by π satisfies the Artin-Rees property as π is central, and so $\widehat{C}$ is right flat over C as C is left Noetherian. Tensoring over R with L, we therefore see that $\widehat{B_L} \cong \widehat{C_L}$ is right flat over $\widehat{A_L} = C_L$. □

3.3 A PBW theorem for $U^{\pm}$

In order to apply the previous results to $\widehat{U_q}$, it will be useful to find certain bases of the algebras U_n. These will in turn allow us to get an explicit description of $\widehat{U_q}$.

Let $\mathcal{U}$ be the R-submodule of U_q spanned by all monomials $M_{\boldsymbol{r},\boldsymbol{s},\lambda}$, which is free by Corollary 2.4.8. The height filtration on U_q induces a filtration on $\mathcal{U}$. Explicitly, we define $F_i\mathcal{U}$ to be the R-span of the monomials $M_{\boldsymbol{r},\boldsymbol{s},\lambda}$ such that $\mathrm{ht}(M_{\boldsymbol{r},\boldsymbol{s},\lambda}) \leq i$. We want to deform this module and eventually obtain an algebra. For each $n \geq 0$, the R-module $\mathcal{U}_n$ is just the R-span of all $\pi^{n\,\mathrm{ht}(M_{\boldsymbol{r},\boldsymbol{s},\lambda})} M_{\boldsymbol{r},\boldsymbol{s},\lambda}$, or in other words the R-span of the monomials

$$(\pi^{n\,\mathrm{ht}(\beta_1)} F_{\beta_1})^{r_1} \cdots (\pi^{n\,\mathrm{ht}(\beta_N)} F_{\beta_N})^{r_N} K_\lambda (\pi^{n\,\mathrm{ht}(\beta_1)} E_{\beta_1})^{s_1} \cdots (\pi^{n\,\mathrm{ht}(\beta_N)} E_{\beta_N})^{s_N}.$$

We let m be the least integer such that

$$\frac{\pi^{2m}}{q_i - q_i^{-1}} \in R \quad \text{for all } 1 \leq i \leq n.$$

Hence for all $n \geq m$ and all i, we have

$$(\pi^n E_{\alpha_i})(\pi^n F_{\alpha_i}) - (\pi^n F_{\alpha_i})(\pi^n E_{\alpha_i}) \in R[K_\lambda : \lambda \in P]$$

and so we see that the presentation of U_q gives rise to relations between the generators of U_n which are well-defined in U_n.

Theorem 3.3.1. *The R-module $\mathcal{U}_n$ is equal to U_n for all $n \geq m$, and so is an R-algebra.*

We start preparing for the proof the Theorem. For all $n \geq 0$, we let U_n^+ be the positive part of U_n, i.e. the R-subalgebra of U_q generated by the $\pi^n E_{\alpha_i}$'s. It is the n-th deformation of U^+ with respect to the filtration given by assigning every E_{α_i} degree 1. We also define $\mathcal{U}_n^+$ to be the R-submodule of $\mathcal{U}_n$ spanned by all monomials of the form

$$(\pi^{n\,\mathrm{ht}(\beta_1)} E_{\beta_1})^{s_1} \cdots (\pi^{n\,\mathrm{ht}(\beta_N)} E_{\beta_N})^{s_N}.$$

It is the n-th deformation of $\mathcal{U}^+$ with respect to the height filtration. We then define U_n^- and $\mathcal{U}_n^-$ by applying ω to the positive parts. We also let $U^0 = \mathcal{U}^0 = R[K_\lambda : \lambda \in P]$.

By our assumptions on $p = \mathrm{char}(k)$, we see that the braid group action from Theorem 2.4.5 preserves U and so E_{β_j} lies in U for all $1 \leq j \leq N$. Since the automorphism ω from Proposition 2.4.2(i) preserves U, we see that the F_{β_j}'s also belong to U, and hence that $\mathcal{U} \subset U$. To obtain that the $\pi^{n \operatorname{ht} \beta_j} E_{\beta_j}$'s actually lie in U_n^+ for every $n \geq 0$, we adapt [45, Lemma 8.19 and Proposition 8.20] to our situation. The same proofs go through with only minor changes. Before that, we establish the following notation: for a sequence $J = \{\alpha_{i_1}, \ldots, \alpha_{i_j}\}$ of simple roots, we write E_J for the product $E_{\alpha_{i_1}} \cdots E_{\alpha_{i_j}}$.

Lemma 3.3.2. *Let $w \in W$ and α be a simple root. Suppose $w\alpha > 0$ and write $w\alpha = \sum_{i=1}^n m_i \alpha_i$. Then $T_w(E_\alpha)$ is an R-linear combination of words all of the form E_J where J is a finite sequence of simple roots such that each root α_i occurs in J with multiplicity m_i.*

Proof. We first prove the result in a particular case.

Claim. Suppose $\beta \neq \alpha$ is another simple root and assume w is in the subgroup of W generated by s_α and s_β. Then the result holds.

Proof of claim. We are reduced to a rank 2 case-by-case analysis. If $w = 1$ the result is trivial so assume $w \neq 1$. Denote by m the order of $s_\alpha s_\beta$. We have $m = 2, 3, 4$ or 6.

If $m = 2$ then $w = s_\beta$ and $T_w(E_\alpha) = E_\alpha$. If $m = 3$ then

$$w \in \{s_\beta, s_\alpha s_\beta\}.$$

If $m = 4$ then

$$w \in \{s_\beta, s_\alpha s_\beta, s_\beta s_\alpha s_\beta\}.$$

If $m = 6$ then

$$w \in \{s_\beta, s_\alpha s_\beta, s_\beta s_\alpha s_\beta, s_\alpha s_\beta s_\alpha s_\beta, s_\beta s_\alpha s_\beta s_\alpha s_\beta\}.$$

Hence in all cases we see that $T_w(E_\alpha)$ is just one of the root vectors that arise in the PBW basis for the case where $\mathfrak{g}$ has rank 2. The result then follows by the formulae in [31, Appendix,(A1)-(A3)] using our assumptions on p. □

We now use induction on $\ell(w)$. If $\ell(w) = 0$ then $T_w = 1$ and the result is trivial. So assume that $\ell(w) > 0$. Hence there exists a simple root β such that $w\beta < 0$ (and so $\alpha \neq \beta$). By standard facts about Coxeter groups (see [43, 1.10]), we have a decomposition $w = w'w''$ where w'' lies in the subgroup of W generated by s_α and s_β such that $w'\beta > 0$ and $w'\alpha > 0$. Then $\ell(w) = \ell(w') + \ell(w'')$ so that $T_w = T_{w'}T_{w''}$ by Lemma 2.4.6. Moreover since $w\alpha > 0$ and $w\beta < 0$ it follows that $w''\alpha > 0$ and $w''\beta < 0$. In particular $w'' \neq 1$. By the claim we have that $T_{w''}(E_\alpha)$ is an R-linear combination of words all of the form $E_{J''}$ where J'' is a finite sequence of simple roots only involving α and β such that they appear with the appropriate multiplicities. By induction hypothesis, we also have that $T_{w'}(E_\alpha)$ is an R-linear combination of words all of the form $E_{J'}$ where J' is a finite sequence of simple roots each simple root appears in J' with the appropriate multiplicity. Similarly, the analogous statement is true for $T_{w'}(E_\beta)$. Now the result follows since $T_w = T_{w'}T_{w''}$. □

Corollary 3.3.3. *Let $w_0 = s_{i_1} \cdots s_{i_N}$ be our fixed reduced expression for w_0. For any $1 \leq j \leq N$, write $\beta_j = \sum_{i=1}^n m_{ij}\alpha_i$. Then E_{β_j} is an R-linear combination of words all of* *the form E_J where J is a finite sequence of simple roots such that each root α_i occurs in J with multiplicity m_{ij} (and so J has length $ht\beta_j$).*

Proof. Since $\beta_j := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ we can write it as $w\alpha$ where $w = s_{i_1} \cdots s_{i_{j-1}}$ and $\alpha = \alpha_{i_j}$. $\square$

In particular, the Corollary implies that, for all $n \geq 0$, $\pi^{n\,\mathrm{ht}(\beta_j)} E_{\beta_j} \in U_n^+$ for all $1 \leq j \leq N$. Similarly $\pi^{n\,\mathrm{ht}(\beta_j)} F_{\beta_j} \in U_n^-$ for all j. Hence we see that $\mathcal{U}^{\pm} \subseteq U^{\pm}$ and that $\mathcal{U}_n \subseteq U_n$ for all $n \geq 0$.

Remark 3.3.4. Although the proof that $E_{\beta_j} \in U^+$ is well-known, we couldn't find a reference for the result about multiplicities so we included the proofs for that.

The argument to prove Theorem 3.3.1 is the same as in [45, Theorem 8.24], rephrased in our context. We sketch it here. By our choice of m, the multiplication map

$$U_m^- \otimes_R U_m^0 \otimes_R U_m^+ \to U_m$$

is surjective, where $U_m^0 = R[K_\lambda : \lambda \in P] = RP$. Since the left hand side is a lattice inside $U_q^- \otimes_L U_q^0 \otimes_L U_q^+$ and by using the triangular decomposition (Theorem 2.4.4(i)) for U_q, we in fact get that this map is an isomorphism. We also clearly have a triangular decomposition $\mathcal{U}_m \cong \mathcal{U}_m^- \otimes_R \mathcal{U}_m^0 \otimes_R \mathcal{U}_m^+$ where $\mathcal{U}_m^0 = U_m^0$.

Since the automorphism ω preserves U_m, we only have to check that $U_m^+ = \mathcal{U}_m^+$ in order to obtain $U_m = \mathcal{U}_m$. In fact we show that $U^+ = \mathcal{U}^+$ and that this implies that $U_n^+ = \mathcal{U}_n^+$ for every $n \geq 0$.

Proposition 3.3.5 ([45, Proposition 8.22])**.** *Let $w \in W$ and choose a reduced expression $w = s_{j_1} \cdots s_{j_t}$. Denote by $\mathcal{U}^+[w]$ the R-span of all monomials of the form*

$$E_{\beta_1}^{m_1} \cdots E_{\beta_t}^{m_t} \tag{3.2}$$

where $E_{\beta_i} = T_{\alpha_{j_1}} \cdots T_{\alpha_{j_{i-1}}}(E_{\alpha_{j_i}})$ for $1 \leq i \leq t$. Then $\mathcal{U}^+[w]$ depends only on w, not of the choice of reduced expression.

Remark 3.3.6. We remark here that while [45, Proposition 8.22] is not phrased in terms of integral forms, it is however absolutely identical for the De Concini-Kac integral form. The proof boils down from general arguments to certain rank 2 calculations which all take place inside U^+ (see [45, 8.23 & pages 170-172]).

Corollary 3.3.7. *We have $U_n^+ = \mathcal{U}_n^+$ for every $n \geq 0$. Moreover, the height filtration on $\mathcal{U}^+ = U^+$ equals the filtration obtained by assigning every E_{α_i} degree 1.*

Proof. By Proposition 3.3.5 we see that $\mathcal{U}^+ = \mathcal{U}^+[w_0]$ is independent of the choice of reduced expression for w_0. Now, it is well-known that for each $1 \leq i \leq n$, there exists a reduced expression $w_0 = s_{\alpha_{j_1}} s_{\alpha_{j_2}} \cdots s_{\alpha_{j_N}}$ where $j_1 = i$ (see [43, 1.8]). Hence it follows from the definition of $\mathcal{U}^+[w_0]$ that it is preserved under left multiplication by all the generators E_{α_i}. Since these generate U^+ and since $1 \in \mathcal{U}^+$, this implies that $U^+ = \mathcal{U}^+$.

The height filtration on U^+ is an algebra filtration as it is the subspace filtration of an algebra filtration on U_q^+. Since all the E_{α_i}'s have degree 1 in it, it must contain the filtration where we set $\deg(E_{\alpha_i}) = 1$. Corollary 3.3.3 gives the reverse inclusion. Thus we now obtain $U_n^+ = \mathcal{U}_n^+$ by taking the n-th deformation with respect to this filtration. □

Proof of Theorem 3.3.1. Put $n = m$ in the previous Corollary to obtain that $U_m = \mathcal{U}_m$. Moreover, by the same proof as in the previous Corollary, we get that the height filtration on U_m equals the filtration obtained by setting $F_0U_m = R[K_\lambda : \lambda \in P]$ and $\deg(\pi^m E_{\alpha_i}) = \deg(\pi^m F_{\alpha_i}) = 1$. Hence we get that $U_n = \mathcal{U}_n$ for every $n \geq m$ by deforming. □

Remark 3.3.8. We see that the only thing stopping U from being equal to $\mathcal{U}$ is the commutator relations between the E's and the F's, which stop the triangular identity as we wrote it from holding in U. We can fix this slightly by noticing that we have $U \cong U^- \otimes_R F_0U \otimes U^+$ with a slightly different choice of F_0U: we define it to be the R-algebra generated by the K_λ, $\lambda \in P$, and the elements

$$[K_{\alpha_i}; 0]_{q_i} := \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q_i - q_i^{-1}}$$

for all $i \geq 0$. Then $F_0U = U \cap U_q^0$ and we may define an alternative filtration on U given by assigning each E_{α_i} and F_{α_i} degree 1. Just as in the above proofs, this coincides with the subspace filtration of the height filtration.

We can also use Theorem 3.3.1 to get an explicit description of $\widehat{U_{q,n}}$ for $n \geq m$. Indeed we see that as a topological vector space it is given by the series

$$\widehat{U_{q,n}} = \left\{ \sum_{\boldsymbol{r},\boldsymbol{s},\lambda} a_{\boldsymbol{r},\boldsymbol{s},\lambda} M_{\boldsymbol{r},\boldsymbol{s},\lambda} : \left|\pi^{-n\,\mathrm{ht}(M_{\boldsymbol{r},\boldsymbol{s},\lambda})} a_{\boldsymbol{r},\boldsymbol{s},\lambda}\right| \to 0 \text{ as } \mathrm{ht}(M_{\boldsymbol{r},\boldsymbol{s},\lambda}) \to \infty \right\}.$$

The norm on $\widehat{U_{q,n}}$ is then given by

$$\left|\left| \sum_{\boldsymbol{r},\boldsymbol{s},\lambda} a_{\boldsymbol{r},\boldsymbol{s},\lambda} M_{\boldsymbol{r},\boldsymbol{s},\lambda} \right|\right|_n = \sup_{\boldsymbol{r},\boldsymbol{s},\lambda} \left|\pi^{-n\,\mathrm{ht}(M_{\boldsymbol{r},\boldsymbol{s},\lambda})} a_{\boldsymbol{r},\boldsymbol{s},\lambda}\right|.$$

One can then similarly describe $\widehat{U_q}$:

$$\widehat{U_q} = \left\{ \sum_{\boldsymbol{r},\boldsymbol{s},\lambda} a_{\boldsymbol{r},\boldsymbol{s},\lambda} M_{\boldsymbol{r},\boldsymbol{s},\lambda} : \left|\pi^{-n\,\mathrm{ht}(M_{\boldsymbol{r},\boldsymbol{s},\lambda})} a_{\boldsymbol{r},\boldsymbol{s},\lambda}\right| \to 0 \text{ as } \mathrm{ht}(M_{\boldsymbol{r},\boldsymbol{s},\lambda}) \to \infty \text{ for all } n \geq 0 \right\}.$$

Its Fréchet topology is given by all the norms $||\cdot||_n$.

As another application of this PBW theorem we explain an analogy between our definition of $\widehat{U_q}$ and the Arens-Michael envelope of the classical enveloping algebra $\widehat{U(\mathfrak{g}_L)}$, which is the completion of the enveloping algebra $U(\mathfrak{g}_L)$ with respect to all the submultiplicative semi-norms which extend the norm on L.

As a Fréchet space, $\widehat{U_q}$ is the completion of U_q with respect to the norms $||\cdot||_n$ for $n \geq m$, which are the gauge norms on U_q associated to each U_n. The completion of U_q

with respect to the single norm $||\cdot||_n$ is then $\widehat{U_{q,n}}$. For example these norms take the following values:

$$||E_\alpha||_n = ||F_\alpha||_n = |\pi|^{-n}, \quad ||K_\lambda||_n = 1 \quad \text{for all simple root } \alpha \text{ and all } \lambda \in P.$$

We now aim to show that $\widehat{U_q}$ does not actually depend on the choice of such norms. To make this statement precise, we first consider the canonical norm $||\cdot||$ on the Laurent polynomial ring $L[K_\lambda : \lambda \in P] = U_q^0$, namely the gauge norm associated to U^0. Hence we have $||K_\lambda|| = 1$ for all λ in P. Note that the norms $||\cdot||_n$ are all extensions of $||\cdot||$ to U_q. Indeed, this follows from Theorem 3.3.1 as it implies that $U_n \cap U_q^0 = U^0$.

We will now work in a more general context. Let $A \subset B$ be torsion-free, π-adically separated R-algebras, and equip A_L with the gauge norm associated with A. Suppose that $B \cap A_L = A$, where we regard A, A_L and B as subalgebras of B_L. We say that a semi-norm p on B_L is *submultiplicative* if for all $x, y \in B_L$ we have $p(xy) \leq p(x)p(y)$ and $p(1) = 1$.

Proposition 3.3.9. *For A and B as above, suppose that B is generated as a ring by A and a finite set of elements $x_1, \dots, x_m \in B \setminus A$ which normalise A, i.e. $x_iA = Ax_i$ for all i. For each $1 \leq i \leq m$, pick a positive integer d_i, and consider the A-filtration on B given by assigning degree d_i to x_i for each i. Then, for this filtration, $\widehat{B_L}$ is isomorphic to the completion of B_L with respect to all submultiplicative semi-norms which extend the norm on A_L.*

Proof. The filtration gives rise to a family of norms $||\cdot||_n$ on B_L, which are just the gauge norms on B_L associated to each of the deformations B_n. Since the π-adic filtration on B_n is an algebra filtration, it follows that these norms are submultiplicative. Also, the π-adic topology on B_n restricts to the π-adic topology on A for all n because $B \cap A_L = A$, and so these norms extend the norm on A_L. Hence, since $\widehat{B_L}$ is the completion of B_L with respect to the norms $||\cdot||_n$, there is a canonical map $\mathfrak{B} \to \widehat{B_L}$, where $\mathfrak{B}$ denotes the completion of B_L with respect to all submultiplicative semi-norms that extend the norm on A_L.

To show that this map is a topological algebra isomorphism, it suffices to show that given any submultiplicative semi-norm p on B_L that extends the norm on A_L, there is some n such that $p \leq ||\cdot||_n$. This in turn is equivalent to showing that the unit ball

$$B(p; 1) = \{x \in B_L : p(x) \leq 1\}$$

contains the unit ball of B_L with respect to $||\cdot||_n$, i.e. contains B_n for some n. Now note that since p is submultiplicative and as it extends the norm on A_L, we have that $B(p; 1)$ is an R-algebra containing A. Moreover, by definition of (F.), B_n is the subring of B generated by A and the $\pi^{nd_i}x_i$. So we just need to show that there exists an $n \geq 0$ such that $\pi^{nd_i}x_i \in B(p; 1)$ for all i. But that's clearly true since $p(\pi^{nd_i}x_i) = |\pi|^{nd_i}\, p(x_i) \to 0$ as $n \to \infty$ for any i. $\square$

Corollary 3.3.10. *$\widehat{U_q}$ is isomorphic to the completion of U_q with respect to all submultiplicative semi-norms that extend the gauge norm $||\cdot||_{U^0}$ on U_q^0. Also, $\widehat{\mathcal{O}_q}$ is the completion* *of $\mathcal{O}_q$ with respect to all the submultiplicative semi-norms that extend the norm on L.*

Proof. Set $A = R[K_\lambda : \lambda \in P]$ and $B = U_m$ for $\widehat{U_q}$ (note that $B \cap A_L = A$ by Theorem 3.3.1), and $A = R$ and $B = \mathcal{A}_q$ for $\widehat{\mathcal{O}_q}$. The hypotheses of Proposition 3.3.9 are then satisfied, where for $\widehat{U_q}$ we use the fact that the height filtration on U_m is the filtration given by assigning degree 1 to the E's and the F's (see the proof of Theorem 3.3.1). □

3.4 Fréchet-Stein property of $\widehat{U_q}$ and $\widehat{\mathcal{O}_q}$

We can now start applying our techniques to $\widehat{U_q}$.

Lemma 3.4.1. *The R-algebra U_m satisfies conditions (i) and (ii) from Section 3.2.*

Proof. It is clear that $\operatorname{gr}_0 U_m$ is Noetherian since it's a finitely generated commutative R-algebra. Moreover the height filtration on U_m is the subspace filtration of the height filtration on U_q, thus there is a natural embedding $\operatorname{gr} U_m \hookrightarrow U^{(1)}$ where $U^{(1)} := \operatorname{gr} U_q$. Write $U_m^{(1)} := \operatorname{gr} U_m$. This shows that $U_m^{(1)}$ is π-torsion free, thus flat. Moreover since U_m is free it is also π-adically separated. Therefore U_m is a deformable R-algebra. Recall now that we defined in the discussion preceeding Theorem 2.4.10 a $\mathbb{Z}_{\geq 0}^{2N}$-filtration on $U^{(1)}$. Using the above embedding, we may now give to $U_m^{(1)}$ the corresponding $\mathbb{Z}_{\geq 0}^{2N}$-filtration. We see from the relations in Theorem 2.4.10 that the associated graded algebra of $U_m^{(1)}$ is then q-commutative, hence Noetherian by Lemma 2.5.6. Therefore $U_m^{(1)}$ is Noetherian, and condition (i) is satisfied. Condition (ii) just follows from definition of the height filtration: the generators x_i are just the root elements $\pi^{m\,\mathrm{ht}(\beta_i)} E_{\beta_i}$ and $\pi^{m\,\mathrm{ht}(\beta_i)} F_{\beta_i}$. □

Remark 3.4.2. If we equip U with the filtration from Remark 3.3.8, it is then also true that it satisfies conditions (i) and (ii) and then the same proof as in the Lemma applies. However the Fréchet completion $\widehat{U_L}$ that one gets that way is not the same as $\widehat{U_q}$. Specifically, the norms defining $\widehat{U_L}$ for this filtration all have value 1 at the elements $[K_{\alpha_i}; 0]$, which is not true in $\widehat{U_q}$. Now the triples $(E_{\alpha_i}, F_{\alpha_i}, [K_{\alpha_i}; 0])$ correspond under specialisation at 1 to the usual $\mathfrak{sl}_2$ triples (e_i, f_i, h_i) (for the simple roots) in $\mathfrak{g}_L$, and in the Arens-Michael envelope $\widehat{U(\mathfrak{g}_L)}$, the defining norms do not necessarily have value 1 at h_i. While we are not working with a truly generic quantum group, this analogy motivates our choice of working with $\widehat{U_q}$. Note however that the theorem below is also true, with essentially the same proof, for $\widehat{U_L}$.

Before getting to the next result, we introduce some notation. Let $e_{\alpha_1}, \ldots, e_{\alpha_n}$ be the simple root vectors coming from the Serre presentation of $\mathfrak{g}_L$, which can then be extended to a Chevalley basis $e_{\beta_1}, \ldots, e_{\beta_N}$ of $\mathfrak{n}_L$. We then have that the R-span $\mathfrak{n}_R$ of $e_{\beta_1}, \ldots, e_{\beta_N}$ is a Lie lattice in $\mathfrak{n}$, i.e. a free lattice that is also an R-Lie algebra. Indeed, we have the relations

$$[e_\alpha, e_\beta] = c_{\alpha\beta} e_{\alpha+\beta} \tag{3.3}$$

if $\alpha + \beta$ is a root, where $c_{\alpha\beta} \in \mathbb{Z}$, and $[e_\alpha, e_\beta] = 0$ otherwise (see [42, Theorem 25.2]). We write $\mathfrak{n}_k := \mathfrak{n}_R / \pi \mathfrak{n}_R$, a nilpotent k-Lie algebra.

We let $U(\mathfrak{n}_R)$ be the universal enveloping algebra of $\mathfrak{n}_R$. For $n \geq 0$, we denote by $U(\mathfrak{n}_R)_n$ the R-subalgebra of $U(\mathfrak{n}_R)$ generated by all $\pi^n e_{\alpha_i}$. It is the n-th deformation of $U(\mathfrak{n}_R)$ with respect to the height filtration (which is not the same as the PBW filtration – it is defined completely analogously as the height filtration on U_q). Moreover, $U(\mathfrak{n}_R)_n$ is also the universal enveloping algebra of the R-Lie subalgebra of $\mathfrak{n}_R$ generated by all $\pi^n e_{\alpha_i}$. However, since the relations (3.3) are homogeneous with respect to the height, we see that this R-Lie subalgebra is canonically isomorphic to $\mathfrak{n}_R$ as an R-Lie algebra, by mapping $\pi^n e_{\alpha_i} \to e_{\alpha_i}$, and hence there is a canonical isomorphism of R-algebras $U(\mathfrak{n}_R) \cong U(\mathfrak{n}_R)_n$ for all $n \geq 0$. Thus in particular we have that $U(\mathfrak{n}_R)_n/\pi U(\mathfrak{n}_R)_n \cong U(\mathfrak{n}_k)$. In the light of these facts, we can now prove the following:

Theorem 3.4.3. *$\widehat{U_q}$ is a Fréchet-Stein algebra.*

Proof. By Theorem 3.2.7 and the previous Lemma, the result will follow if we prove that condition (iii) is satisfied in U_n for all $n \geq m$. As before, we let $I = \pi U_n \cap U_{n+1}$. We know that I is generated by π, $\pi^{(n+1)\,\mathrm{ht}\,\beta_i} E_{\beta_i}$ and $\pi^{(n+1)\,\mathrm{ht}\,\beta_j} F_{\beta_j}$ $(1 \leq i, j \leq N)$ by Proposition 3.2.5(ii). Observe that $\overline{\pi^{n+1} E_{\alpha_i}}$ commutes with $\overline{\pi^{n+1} F_{\alpha_j}}$ for all i, j since $\pi^n E_{\alpha_i}$ and $\pi^n F_{\alpha_j}$ commute in $\operatorname{gr} U_n$, and so the same can be said of $\overline{\pi^{(m+1)\,\mathrm{ht}\,\beta_i} E_{\beta_i}}$ and $\overline{\pi^{(m+1)\,\mathrm{ht}\,\beta_j} F_{\beta_j}}$. Moreover we also have that all $\overline{\pi^{(m+1)\,\mathrm{ht}\,\beta_i} E_{\beta_i}}$ and $\overline{\pi^{(m+1)\,\mathrm{ht}\,\beta_j} F_{\beta_j}}$ q-commute with $\overline{K_\lambda}$ for all $\lambda \in P$.

Therefore it is enough to show that the elements $\overline{\pi^{(n+1)\,\mathrm{ht}\,\beta_i} E_{\beta_i}}$ for all i form a polycentral sequence in $U^+_{n+1}/\pi U^+_{n+1}$, since the ideal I is preserved by the automorphism ω. But since $q \equiv 1 \pmod{\pi}$ we have a surjection

$$U(\mathfrak{n}_k) \cong U(\mathfrak{n}_R)_{n+1}/\pi U(\mathfrak{n}_R)_{n+1} \to U^+_{n+1}/\pi U^+_{n+1}$$

from the universal enveloping algebra of $\mathfrak{n}_k$, which sends e_{α_i} to $\overline{\pi^{n+1} E_{\alpha_i}}$. In fact, by considering PBW bases we see that this is an isomorphism. Hence it suffices to show that the elements of the Chevalley basis in some order form a polycentral sequence in $U(\mathfrak{n}_k)$. But that is a well known fact (and more generally any ideal in the enveloping algebra of a nilpotent Lie algebra is polycentral by [79, Theorem A]). □

By applying Corollary 3.2.10 we immediately get:

Corollary 3.4.4. *The natural map $U_q \to \widehat{U_q}$ is flat.*

Remark 3.4.5. The Corollary gives an exact functor $M \mapsto \widehat{U_q} \otimes_{U_q} M$ between the category of finitely generated U_q-modules and the category of coadmissible $\widehat{U_q}$-modules. Indeed, since U_q is Noetherian, a finitely generated module M is finitely presented, hence so is $\widehat{U_q} \otimes_{U_q} M$ as a $\widehat{U_q}$-module. In particular, since a finite dimensional U_q-module V is complete with respect to any norm (see Proposition 2.7.6), we have $V \cong \widehat{U_q} \otimes_{U_q} V$ and so V is also coadmissible as a $\widehat{U_q}$-module.

We now turn to $\widehat{\mathcal{O}_q}$. As an L-algebra, $\mathcal{O}_q$ is generated by by the matrix coefficients of the fundamental representations $x_1, \ldots, x_r$. Now the issue is that the q-commutator relations between these are not necessarily defined over R here. Indeed recall from the relations between the x_i given in equation (2.6) that we have

$$x_i x_j = q_{ij} x_j x_i + \sum_{s=1}^{j-1} \sum_{t=1}^{r} (\alpha_{ij}^{st} x_s x_t + \beta_{ij}^{st} x_t x_s),$$

for $1 \leq j < i \leq r$ with $\alpha_{ij}^{st}, \beta_{ij}^{st} \in L$ for all i, j, s, t. These relations are obtained by considering $\mathcal{R}$-matrices for representations of U_q (see [25, Theorem I.8.16]) and it is unclear to us whether the $\mathcal{R}$-matrices are the same when considering integral forms. Note however that the defining relations of $\mathcal{O}_q$ are defined over R in type A by Example 2.5.9.

We fix this issue by deforming enough. Recall the filtration on $\mathcal{O}_q$ given by assigning to each x_i degree $d_i = 2^r - 2^{r-i}$. This choice of degree has the property that whenever $i > j > s$ and $t \leq r$, we always have $d_i + d_j > d_s + d_t$. Thus we see that if we let $y_i = \pi^{ld_i} x_i$ for l sufficiently large, multiplying the above relation by $\pi^{l(d_i+d_j)}$ yields

$$y_i y_j = q_{ij} y_j y_i + \sum_{s=1}^{j-1} \sum_{t=1}^{r} (\alpha_{ij}'^{st} y_s y_t + \beta_{ij}'^{st} y_t y_s), \tag{3.4}$$

where now $\alpha_{ij}'^{st}, \beta_{ij}'^{st} \in R$. Fix the smallest l such that this holds and let B be the R-subalgebra of $\mathcal{O}_q$ generated by $y_1, \ldots, y_r$.

Lemma 3.4.6. *The algebra B is Noetherian, π-adically separated and π-torsion free.*

Proof. B is π-torsion free and π-adically separated because $\mathcal{A}_q$ is. For the Noetherianity, we adapt the proof of [25, Proposition I.8.17]. Let $(F'_\cdot)$ be the filtration on B given by assigning degree d_i to each y_i. Then whenever $i > j > s$ and $t \leq r$, we always have $d_i + d_j > d_s + d_t$. Now let $z_i = y_i + F_{d_i - 1} \in \operatorname{gr}^{F'} B$ be the symbol of y_i. By definition of the filtration, any homogeneous component of $\operatorname{gr}^{F'} B$ of degree d, if it is non-zero, is spanned by the products $z_{i_1} z_{i_2} \cdots z_{i_r}$ such that $d_{i_1} + d_{i_2} + \cdots + d_{i_r} = d$. So $\operatorname{gr}^{F'} B$ is generated by the z_i.

Moreover, by (3.4), for $i > j$ the element $y_i y_j - q_{ij} y_j y_i$ is an R-linear combination of products $y_s y_t$ and $y_t y_s$ with $s < j$. For such s and t, these products have degree $d_s + d_t < d_i + d_j$. Hence we see that $z_i z_j = q_{ij} z_j z_i$. Therefore, we see that $\operatorname{gr}^{F'} B$ is q-commutative over R and so is Noetherian by Lemma 2.5.6. $\square$

We now filter B by assigning degree 1 to all the y_i's. By Proposition 3.3.9 we see that $\widehat{\mathcal{O}_q} \cong \widehat{B_L}$. Let $A = B_1$ be the first deformation of B, i.e. the R-subalgebra of $\mathcal{O}_q$ generated by $\pi y_1, \ldots, \pi y_r$. Completely analogously as in Lemma 3.4.6, we see that A is Noetherian, π-adically separated and π-torsion free. We now set a new filtration on B by defining

$$G_t B = A \cdot \{ y_{i_1} a_{i_1} \cdots y_{i_l} a_{i_l} : a_{i_j} \in A \text{ and } \sum_{j=1}^{l} d_{i_j} \leq t \}.$$

This is the smallest algebra filtration on B such that $y_i \in G_{d_i} B$ and $A = G_0 B$.

Proposition 3.4.7. *With respect to the above filtration, the associated graded ring $\operatorname{gr}^G B$*

is finitely generated as an A-algebra by elements which q-commute with the R-algebra generators of A, and which also q-commute with each other.

Proof. Set $z_i := y_i + G_{d_i-1}B \in \operatorname{gr}^G B$ to be the symbol of y_i for each $1 \le i \le r$. Any homogeneous component $\operatorname{gr}^G_t B$, if it is non-zero, is spanned over A by the symbols of the products $y_{i_1}a_{i_1}\cdots y_{i_l}a_{i_l}$ such that $\sum_{j=1}^{l} d_{i_j} = t$, and any such element equals $z_{i_1}a_{i_1}\cdots z_{i_l}a_{i_l}$. Therefore $\operatorname{gr}^G B$ is generated over A by the z_i.

Now, for any $1 \le j < i \le r$, we have

$$\begin{aligned} y_i(\pi y_j) - q_{ij}(\pi y_j)y_i &= (\pi y_i)y_j - q_{ij}y_j(\pi y_i) \\ &= \sum_{s=1}^{j-1}\sum_{t=1}^{r}\left(\alpha'^{st}_{ij}y_s(\pi y_t) + \beta'^{st}_{ij}(\pi y_t)y_s\right) \in G_{d_j-1}B. \end{aligned}$$

Therefore we see that $z_i(\pi y_j) = q_{ij}(\pi y_j)z_i$ in $\operatorname{gr}^G B$ for all i,j, so that the z_i's will q-commute with the generators of A. Furthermore we have $z_iz_j = q_{ij}z_jz_i$, i.e. the z_i's will q-commute with each other in $\operatorname{gr}_G B$. Indeed, as in the proof of Lemma 3.4.6, this follows from the relations between y_i and y_j in (3.4) because the d_i's were chosen so that whenever $i > j > s$ we have for any $1 \le t \le r$ that $d_i + d_j > d_s + d_t$. □

Theorem 3.4.8. *$\widehat{\mathcal{O}_q}$ is a Fréchet-Stein algebra.*

Proof. By Proposition 3.2.11, it follows from the previous Proposition that $\widehat{B_L}$ is right flat over $\widehat{A_L}$ and that they are both left Noetherian. Left flatness and right Noetherianity will follow by the same argument applied to B^{op}. Thus we see that $\widehat{B_L}$ is flat over $\widehat{A_L}$. For any $n \ge 1$, we can repeat the entire above arguments replacing B by the R-algebra generated by $\pi^n y_i$ for all i, and A by the R-algebra generated by $\pi^{n+1}y_i$ for all i. □

This completes the proof of Theorem A.

3.5 Verma modules and category $\hat{\mathcal{O}}$ for $\widehat{U_q}$

We now start discussing an analogue of category $\mathcal{O}$ for $\widehat{U_q}$. Most of the content of this Section is inspired by [72], and in fact most of our arguments work identically to there.

We begin with a discussion of semisimplicity for modules over the algebras $\widehat{U_q^0} := \widehat{U^0} \otimes_R L$ and $\widehat{(U^{\mathrm{res}})^0_L}$. To simplify notation, we will let $\mathcal{H}$ denote both of these when the proofs are identical for either of them. Our treatment is inspired by the work of Féaux de Lacroix [37].

We will consider the category $\mathscr{M}(\mathcal{H})$ whose objects are Fréchet spaces $\mathcal{M}$ endowed with an action of $\mathcal{H}$ by continuous L-linear endomorphisms, and whose morphisms are continuous L-linear maps which preserve the action of $\mathcal{H}$. Given an object $\mathcal{M}$ of this category and $\lambda \in P$, we denote by $\mathcal{M}_\lambda$ the λ-weight space of $\mathcal{M}$ when viewed as a U_q^0-module.

Definition 3.5.1. We say that $\mathcal{M}$ as above is *topologically $\mathcal{H}$-semisimple* if for every $m \in \mathcal{M}$ there exists a family $\{m_\lambda \in \mathcal{M}_\lambda\}_{\lambda\in P}$ such that $\sum_{\lambda\in P} m_\lambda$ converges to m in $\mathcal{M}$.

We want to investigate the full subcategory $\mathcal{D}(\mathcal{H})$ of $\mathscr{M}(\mathcal{H})$ whose objects are the topologically $\mathcal{H}$-semisimple modules. We first need a couple of preparatory results.

We identify the weight lattice P with its image in T_P, which itself can be identified with the group of characters of U_q^0. Let $x \in U_q^0$. For every $\lambda \in P$ we write $x(\lambda) = \psi_\lambda(x) \in L$. Note that if $x \in (U^{\text{res}})^0$ or U^0, then $x(\lambda) \in R$ for all $\lambda \in P$. Let $q' = q^{1/d}$ so that $q^{\langle \lambda, \mu \rangle} \in (q')^{\mathbb{Z}}$ for any $\lambda, \mu \in P$.

Lemma 3.5.2. *Let $r \in \mathbb{N}$, $m_1, \ldots, m_r \in \mathbb{Z}$ and $\omega_1, \ldots, \omega_r$ be (not necessarily distinct) fundamental weights. For each $\gamma \in P$, write $n_i(\gamma) = d\langle \gamma, \omega_i \rangle \in \mathbb{Z}$ and let*

$$P_\gamma(t) = \prod_{i=1}^{r} (t^{n_i(\gamma)} - (q')^{m_i}) \in R[t, t^{-1}].$$

Then, for every positive integer $a \geq 1$, the image of the set $\{P_\gamma(q') : \gamma \in P\}$ in $R/\pi^a R$ is finite.

Proof. First let $b = v_\pi(q' - 1) > 0$ and note that $b = v_\pi((q')^{-1} - 1)$. Consider

$$Q_\gamma(t) = \prod_{i=1}^{r} (t^{n_i(\gamma) - m_i} - 1) \in R[t, t^{-1}].$$

Then we see that $P_\gamma(q') = (q')^{m_1 + \cdots + m_r} Q_\gamma(q') \equiv Q_\gamma(q') \pmod{\pi}$ since $q' \equiv 1 \pmod{\pi}$, so that it suffices to show that the result holds for $Q_\gamma(t)$. Note that since $v_\pi((q')^m - 1) \geq b\,|m|$ for any $m \in \mathbb{Z}$, it follows that $Q_\gamma(q') \equiv 0 \pmod{\pi^a}$ whenever $b\,|n_i(\gamma) - m_i| \geq a$ for any $1 \leq i \leq r$. Let

$$X = \{(k_1, \ldots, k_r) \in \mathbb{Z}^r : b\,|k_i| < a \text{ for all } 1 \leq i \leq r\}$$

and set

$$M = \left\{ \prod_{i=1}^{r} ((q')^{k_i} - 1) : (k_1, \ldots, k_r) \in X \right\} \cup \{0\}.$$

Then by the above observation we have that every $Q_\gamma(q')$ is congruent to an element of M modulo π^a. The result follows since M is finite. □

Proposition 3.5.3. *Suppose that X is a finite subset of P and let $\lambda \in P \setminus X$. Then there is an element $p \in U_q^0$ such that $p(P) \subset R$, $p(X) = 0$ and $p(\lambda) = 1$.*

Proof. For each $\mu \in X$, the character ψ_μ is determined by its action on the K_{ϖ_i}, so as $\lambda \neq \mu$ there must be some $h_\mu \in \{K_{\varpi_1}, \ldots, K_{\varpi_n}\}$ such that $h_\mu(\lambda) \neq h_\mu(\mu)$. Consider the product

$$x = \prod_{\mu \in X} (h_\mu - h_\mu(\mu)) \in U^0.$$

Note that $h_\mu(P) \subset R$ for every $\mu \in X$ and that, furthermore, the image of $h_\mu(P)$ in $k = R/\pi R$ is constant equal to 1 because $K_{\varpi_i}(\gamma) = q^{\langle \gamma, \varpi_i \rangle} \equiv 1 \pmod{\pi}$ for any $1 \leq i \leq n$ and any $\gamma \in P$. So $x(X) = 0$, $x(\lambda) \neq 0$ and $x(P) \subset R$, actually such that $x(P)$ has image zero in k. Hence there exists a maximal $N > 0$ such that $y := \pi^{-N} x$ still satisfies $y(P) \subset R$, and of course we still have $y(X) = 0$ and $y(\lambda) \neq 0$.

Now note that if $y(\lambda) \in R^\times$, then $p = y(\lambda)^{-1}y$ satifies the required hypothesis. Otherwise, note that the set of residues of $y(P)$ in $R/\pi^a R$ is in bijection with the residues of $x(P) = \pi^N y(P)$ in $R/\pi^{N+a}R$, hence is finite for any $a \geq 1$ by Lemma 3.5.2. Let V be a finite set in R, containing 0, such that every element of $y(P)$ is congruent to a unique element of V modulo π, and set

$$g = \pi^{-1} \prod_{v \in V} (t - v) \in L[t].$$

Then $g(y(P)) \subset R$, $g(y(X)) = 0$ and $v_\pi(g(y(\lambda))) = v_\pi(y(\lambda)) - 1$. Moreover the image of $g(y(P))$ in $R/\pi^a R$ is in bijection with the image of $\pi g(y(P))$ in $R/\pi^{a+1}R$, which is finite for every $a \geq 1$ since it was for $y(P)$. By induction, we can then find $h \in L[t]$ such that $p := h(g(y))$ satisfies the required properties. $\square$

Theorem 3.5.4. *Suppose that $\mathcal{M} \in \mathcal{D}(\mathcal{H})$. Then for each $m \in \mathcal{M}$, there exists a unique family $(m_\lambda)_{\lambda \in P}$ with $m_\lambda \in \mathcal{M}_\lambda$ such that $\sum_{\lambda \in P} m_\lambda$ converges to m. Moreover, if $m \in \mathcal{N}$ where $\mathcal{N}$ is a closed U_q^0-invariant subspace, then each $m_\lambda \in \mathcal{N}$.*

Proof. We know by definition that there is a family $(m_\lambda)_{\lambda \in P}$ with $m_\lambda \in \mathcal{M}_\lambda$ such that $\sum_{\lambda \in P} m_\lambda$ converges to m. So we just need to prove uniqueness. Fix $\mu \in P$, and let $q_1 \leq q_2 \leq \cdots$ be a countable set of semi-norms defining the topology on $\mathcal{M}$, so that $\mathcal{M} \cong \varprojlim \mathcal{M}_{q_i}$.

Fix some $i \geq 1$. There is an ascending chain $S_1 \subset S_2 \subset \cdots$ of finite subsets of P such that $\lambda \in P \setminus S_j$ implies that $q_i(m_\lambda) \leq 1/j$. By Proposition 3.5.3, for every $j \geq 1$, there exists $p_j \in U_q^0$ such that $p_j(P) \subset R$, $p_j(S_j \setminus \{\mu\}) = 0$ and $p_j(\mu) = 1$. Then we have

$$p_j \cdot m = \sum_{\lambda \in P} p_j(\lambda) m_\lambda = m_\mu + \sum_{\lambda \in P \setminus S_j} p_j(\lambda) m_\lambda.$$

By construction, $q_i(p_j(\lambda)m_\lambda) \leq q_i(m_\lambda) \leq 1/j$ for all $\lambda \in P \setminus S_j$. Hence $p_j \cdot m \to m_\mu$ in $\mathcal{M}_{q_i}$ as $j \to \infty$. So we see that the image of m_μ in $\mathcal{M}_{q_i}$ is uniquely determined by m by uniqueness of limits. Since i was arbitrary and since $\mathcal{M} \cong \varprojlim \mathcal{M}_{q_i}$, it follows that m_μ is uniquely determined by m.

For the last part, since $\mathcal{N}$ is closed and so complete, it follows that $\mathcal{N}_{q_i}$ is equal to the closure of $\mathcal{N}$ in $\mathcal{M}_{q_i}$ for each $i \geq 1$, and $\mathcal{N} \cong \varprojlim \mathcal{N}_{q_i}$. Now $\mathcal{N}$ is U_q^0-invariant. So for every $i \geq 1$ we have that the image of m_μ in $\mathcal{M}_{q_i}$ equals $\lim p_j \cdot m \in \mathcal{N}_{q_i}$. Hence $m_\mu \in \mathcal{N}$. $\square$

Remark 3.5.5. The ideas in the proofs of Theorem 3.5.4 and Proposition 3.5.3 were adapted for quantum groups from a proof that was communicated to us privately by Simon Wadsley.

Given $\mathcal{M} \in \mathcal{D}(\mathcal{H})$, we may form

$$M^{\mathrm{ss}} = \bigoplus_{\lambda \in P} M_\lambda$$

which is a U_q^0-module. From the above, we immediately get the first part of the next result:

Corollary 3.5.6. *The category $\mathcal{D}(\mathcal{H})$ is stable under passage to closed $\mathcal{H}$-submodules and to the corresponding quotients. Moreover, given $\mathcal{M} \in \mathcal{D}(\mathcal{H})$ and a closed submodule $\mathcal{N}$, we have $(\mathcal{M}/\mathcal{N})^{\mathrm{ss}} \cong \mathcal{M}^{\mathrm{ss}}/\mathcal{N}^{\mathrm{ss}}$.*

Proof. For the last part, for every $m \in \mathcal{M}$, write $\overline{m}$ for its image in the quotient $\mathcal{M}/\mathcal{N}$. Suppose that $\overline{m} \in (\mathcal{M}/\mathcal{N})^{\mathrm{ss}}$. By continuity of the quotient map, if $m = \sum_{\lambda \in P} m_\lambda$ converges then $\overline{m} = \sum_{\lambda \in P} \overline{m_\lambda}$ converges too, and that sum must be finite by the uniqueness of the decomposition from Theorem 3.5.4. Thus there is a finite set $S \subset P$ such that, if $\lambda \in P \setminus S$, then $m_\lambda \in \mathcal{N}$. Hence if we write $m' = \sum_{\lambda \in S} m_\lambda \in \mathcal{M}^{\mathrm{ss}}$, then $\overline{m'} = \overline{m}$. This shows that the map

$$\mathcal{M}^{\mathrm{ss}} \to (\mathcal{M}/\mathcal{N})^{\mathrm{ss}}$$

is surjective. We now simply observe that its kernel is $\mathcal{N}^{\mathrm{ss}}$. □

Proposition 3.5.7. *Suppose that $\mathcal{M} \in \mathcal{D}(\mathcal{H})$. Then the assignement*

$$f : \mathcal{N} \mapsto \mathcal{N} \cap \mathcal{M}^{ss}$$

defines an injective map between the set of closed $\mathcal{H}$-submodules of $\mathcal{M}$ and the set of abstract U_q^0-submodules of $\mathcal{M}^{ss}$, with left inverse given by passing to the closure in $\mathcal{M}$. Now assume furthermore that all the weight spaces $\mathcal{M}_\lambda$ are finite dimensional. Then f is in fact surjective and so bijective. If additionally, $\mathcal{M}$ is also equipped with a U_q-action by continuous L-linear endomorphisms extending the U_q^0-action, then the bijection descends to a bijection between the U_q-invariant objects.

Proof. The proof we give is completely analogous to [37, Satz 1.3.19 & Kor. 1.3.22]. For the first part, we must show that $\mathcal{N} = \overline{\mathcal{N} \cap \mathcal{M}^{\mathrm{ss}}}$. Pick $m \in \mathcal{N}$. By Theorem 3.5.4, we may write $m = \sum_{\lambda \in P} m_\lambda$ where $m_\lambda \in \mathcal{N}$ for each $\lambda \in P$. For each $n \in \mathbb{N}$, let

$$P_n = \left\{ \sum n_i \varpi_i \in P : |n_i| \leq n \right\}.$$

Since each P_n is a finite set, we may define $m_n = \sum_{\lambda \in P_n} m_\lambda \in \mathcal{N} \cap \mathcal{M}^{\mathrm{ss}}$. Then we have $m_n \to m$ as $n \to \infty$ and so $m \in \overline{\mathcal{N} \cap \mathcal{M}^{\mathrm{ss}}}$. Thus we see that $\mathcal{N} \subseteq \overline{\mathcal{N} \cap \mathcal{M}^{\mathrm{ss}}}$. The other inclusion is trivial.

Now assume all weight spaces are finite dimensional, and let $N \subseteq \mathcal{M}^{\mathrm{ss}}$ be a U_q^0-submodule. Note that N must be semisimple since $\mathcal{M}^{\mathrm{ss}}$ is semisimple. The result will follow if we show that for such an N, we always have $N = \overline{N} \cap \mathcal{M}^{\mathrm{ss}}$. To do that, we need to show that $\overline{N} \cap \mathcal{M}^{\mathrm{ss}}$ is contained in N, the other inclusion being clear. So pick $m \in \overline{N} \cap \mathcal{M}^{\mathrm{ss}}$. Then there exists $(m_j)_{j \in \mathbb{N}}$ such that $m_j \in N$ for all j and $m_j \to m$ as $j \to \infty$. Since all the m_j lie in $\mathcal{M}^{\mathrm{ss}}$, we can find an ascending chain of finite subsets $S_j \subseteq P$ such that $m_j = \sum_{\lambda \in S_j} m_{\lambda,j}$ with $m_{\lambda,j} \in \mathcal{M}_\lambda$. We may also find a finite subset $S_0 \subseteq P$ such that $m = \sum_{\lambda \in S_0} m_\lambda$ with $m_\lambda \in \mathcal{M}_\lambda$, and without loss of generality we may assume that $S_0 \subseteq S_1$. Let $S = \bigcup_{j \geq 0} S_j$.

Now it follows from our assumption on weight spaces that any finite direct sum of weight spaces is finite dimensional, and the subspace topology on it is equivalent to the

Banach space topology given by the max norm (see Proposition 2.7.6). In particular the projection map to any direct summand is continuous. Since $\mathcal{M}^{\mathrm{ss}}$ is the direct limit of the these finite direct sums, we see that the projection map from $\mathcal{M}^{\mathrm{ss}}$ to any direct summand is continuous, where $\mathcal{M}^{\mathrm{ss}}$ is given the subspace topology. Hence we have that, for a fixed $\lambda \in S$, $m_{\lambda,j}$ converges to m_λ (where $m_{\lambda,j}$, respectively m_λ, is understood to be zero when $\lambda \notin S_j$, respectively $\lambda \notin S_0$). But now $m_{\lambda,j} \in N \cap \mathcal{M}_\lambda$ for every j, and $N \cap \mathcal{M}_\lambda$ is finite dimensional hence complete again by Proposition 2.7.6. So we get that $m_\lambda \in N$ for every $\lambda \in S_0$ as required.

For the last part, we have that $\mathcal{M}^{\mathrm{ss}}$ is then a U_q-submodule of $\mathcal{M}$, so that $\mathcal{N} \cap \mathcal{M}^{\mathrm{ss}}$ is U_q-invariant whenever $\mathcal{N}$ is U_q-invariant. Also U_q-invariant subspaces of $\mathcal{M}$ are preserved under passing to the closure. Hence the result follows immediately from the above. $\square$

We are now in a position where we can define an analogue of the BGG category $\mathcal{O}$ for $\widehat{U_q}$. First we recall that there is a category, that we denote by $\mathcal{O}$, which is the full subcategory of the category of U_q-modules consisting of modules M that satisfy the following:

- M is finitely generated;
- M is the sum of its weight spaces (of type $\mathbf{1}$); and
- $\dim_L U_q^+ m < \infty$ for all $m \in M$.

This category is an analogue of the integral subcategory $\mathcal{O}_{\mathrm{int}}$ (i.e. the direct sum of all integral blocks) of the usual BGG category $\mathcal{O}$ for the complex Lie algebra $\mathfrak{g}$ (see [44]). Our category $\mathcal{O}$ shares all the standard properties of $\mathcal{O}_{\mathrm{int}}$, see [3, Section 6] and [27, Chapters 9-10]. In particular, all modules in $\mathcal{O}$ have finite dimensional weight spaces and have finite length, the highest weight U_q-modules all belong to that category, are indecomposable and have a unique simple quotient, and $\mathcal{O}$ splits into blocks

$$\mathcal{O} = \bigoplus_{\lambda \in -\rho + P^+} \mathcal{O}^\lambda$$

where ρ is half the sum of the positive roots, and the block $\mathcal{O}^\lambda$ consists of those modules from $\mathcal{O}$ whose composition factors have highest weights in $W \cdot \lambda$.

Now we have for each $n \geq m$ that $U^0 = R[K_\lambda : \lambda \in P] \subset U_n$ and from the PBW theorem (Theorem 3.3.1) we see that $\pi^a U_n \cap U^0 = \pi^a U^0$ for every $a \geq 1$. Hence it follows that the subspace topology on U^0 of the π-adic topology on U_n is the π-adic topology on U^0. Thus we see that the injection $U_q^0 \subseteq U_q$ is strict (in fact an isometry) with respect to all the norms $||\cdot||_n$ for $n \geq m$ on U_q and the single gauge norm $||\cdot||$ on U_q^0 associated to U_q^0. Hence there is a canonical strict embedding $\widehat{U_q^0} \hookrightarrow \widehat{U_q}$.

Moreover, recall from the notion of a coadmissible module from Definition 3.2.8 and the properties of the category $\mathcal{C}(\widehat{U_q})$ from Proposition 3.2.9. These modules have a Fréchet topology attached to them, making them by the above into $\widehat{U_q^0}$-modules where the action is by continuous L-linear endomorphisms.

Definition 3.5.8. The category $\hat{\mathcal{O}}$ for $\widehat{U_q}$ is defined to be the full subcategory of $\mathcal{C}(\widehat{U_q})$ consisting of coadmissible modules $\mathcal{M}$ satisfying:

(i) $\mathcal{M}$ is topologically $\widehat{U_q^0}$-semisimple with weights contained in finitely many cosets of the form $\lambda - Q^+$, with $\lambda \in P$; and

(ii) All weight spaces of $\mathcal{M}$ are finite dimensional.

From Proposition 3.2.9 and Corollary 3.5.6, we immediately get:

Proposition 3.5.9. *Let $\mathcal{M}$ be an object of $\hat{\mathcal{O}}$.*

(i) The direct sum of two objects in $\hat{\mathcal{O}}$ is in $\hat{\mathcal{O}}$;

(ii) the category $\hat{\mathcal{O}}$ is an abelian subcategory of $\mathcal{C}(\widehat{U_q})$;

(iii) the sum of two coadmissible submodules of $\mathcal{M}$ is in $\hat{\mathcal{O}}$;

(iv) any finitely generated submodule of $\mathcal{M}$ is in $\hat{\mathcal{O}}$; and

(v) Let $\mathcal{N}$ be a submodule of $\mathcal{M}$. Then the following are equivalent:

(1) $\mathcal{N}$ is in $\hat{\mathcal{O}}$;

(2) $\mathcal{M}/\mathcal{N}$ is in $\hat{\mathcal{O}}$; and

(3) $\mathcal{N}$ is closed in the Fréchet topology of $\mathcal{M}$.

We also record here the following fact:

Lemma 3.5.10. *Let $\mathcal{M} \in \hat{\mathcal{O}}$. There is an inclusion preserving bijection between the subobjects of $\mathcal{M}$ in $\hat{\mathcal{O}}$ and the U_q-submodules of $\mathcal{M}^{\mathrm{ss}}$.*

Proof. We see from Proposition 3.5.7 that the map

$$\mathcal{N} \mapsto \mathcal{N} \cap \mathcal{M}^{\mathrm{ss}}$$

gives an inclusion preserving bijection between the closed, U_q-invariant, $\widehat{U_q^0}$-submodules of $\mathcal{M}$ and the U_q-submodules of $\mathcal{M}^{ss}$. But the former are just the closed $\widehat{U_q}$-submodules of $\mathcal{M}$, which are just the subobjects in $\hat{\mathcal{O}}$ by Proposition 3.5.9(v). □

We may now define the objects which play the role of Verma modules. For each $\lambda \in P$, we let I_λ be the left ideal of $\widehat{U_q}$ generated by all E_{α_i}, $K_{\varpi_i} - \lambda(K_{\varpi_i})$ $(1 \leq i \leq n)$. Since it is finitely generated, it must be a coadmissible module and hence the quotient $\widehat{U_q}/I_\lambda$ is coadmissible as well.

Definition 3.5.11. We define the *Verma module with highest weight* λ for $\widehat{U_q}$ to be the quotient $\widehat{M_\lambda} := \widehat{U_q}/I_\lambda$, which is a coadmissible module.

Note that by flatness of $\widehat{U_q}$ as a U_q-module (Corollary 3.4.4), we get that $\widehat{M_\lambda} \cong \widehat{U_q} \otimes_{U_q} M_\lambda$. Indeed, if J_λ denotes the left ideal of U_q generated by all E_{α_i}, $K_{\varpi_i} - \lambda(K_{\varpi_i})$ $(1 \leq i \leq n)$, then we have a short exact sequence

$$0 \to J_\lambda \to U_q \to M_\lambda \to 0$$

of U_q-modules, and our claim follows by tensoring it with $\widehat{U_q}$.

We now want to show that $\widehat{M_\lambda}$ is an object of our category. To do this, we will need a tensor product decomposition of $\widehat{U_q}$. Consider the filtration on U^- given by assigning each F_{α_i} degree 1 (this is the same as the height filtration by Corollary 3.3.7). The n-th deformation of U^- with respect to this filtration is just U_n^- for each $n \geq 0$. For $n \geq m$, by the PBW theorem (Theorem 3.3.1), we have that $\pi^a U_n \cap U_n^- = \pi^a U_n^-$ for every $a \geq 0$, so that there is an isometric embedding

$$\widehat{U_{q,n}^-} := \widehat{U_n^-} \otimes_R L \hookrightarrow \widehat{U_{q,n}}.$$

Hence if we let $\widehat{U_q^-} := \varprojlim \widehat{U_{q,n}^-}$, then there is a strict embedding $\widehat{U_q^-} \hookrightarrow \widehat{U_q}$. Using Corollary 3.3.7, we may describe $\widehat{U_q^-}$ explicitly as follows:

$$\widehat{U_q^-} = \left\{ \sum_{\boldsymbol{r}} a_{\boldsymbol{r}} F_{\beta_1}^{r_1} \cdots F_{\beta_N}^{r_N} : \left| \pi^{-n \operatorname{ht}(F^{\boldsymbol{r}})} a_{\boldsymbol{r},\boldsymbol{s},\lambda} \right| \to 0 \text{ as } \operatorname{ht}(F^{\boldsymbol{r}}) \to \infty \text{ for all } n \geq 0 \right\}. \quad (3.5)$$

We may completely analogously define the positive subalgebra of $\widehat{U_q}$.

We can also do a similar construction for the positive Borel. For each $n \geq m$, the inclusion $U_n^{\geq 0} \subseteq U_n$ induces an isometric embedding

$$\widehat{U_{q,n}^{\geq 0}} := \widehat{U_n^{\geq 0}} \otimes_R L \hookrightarrow \widehat{U_{q,n}}$$

and passing to the inverse limit, this gives a strict embedding $\widehat{U_q^{\geq 0}} \hookrightarrow \widehat{U_q}$ where $\widehat{U_q^{\geq 0}} = \varprojlim \widehat{U_{q,n}^{\geq 0}}$.

Lemma 3.5.12. *The multiplication map defines a topological isomorphism*

$$\widehat{U_q^-} \widehat{\otimes}_L \widehat{U_q^{\geq 0}} \to \widehat{U_q}$$

of bimodules.

Proof. The PBW theorem (Theorem 3.3.1) for U_m gives an isomorphism

$$U_m^- \otimes_R U_m^{\geq 0} \cong U_m$$

of filtered R-modules. The result follows from Theorem 3.1.8. $\square$

Note that, for every $\lambda \in P$, the one dimensional $U_q^{\geq 0}$-module L_λ, being complete with respect to any locally convex topology, naturally extends to a $\widehat{U_q^{\geq 0}}$-module.

Proposition 3.5.13. *The module $\widehat{M_\lambda}$ lies in $\hat{\mathcal{O}}$ and $\widehat{M_\lambda}^{\text{ss}} = M_\lambda$. There is a canonical inclusion preserving bijection between the subobjects of $\widehat{M_\lambda}$ and the U_q-submodules of M_λ. In particular, $\widehat{M_\lambda}$ is an irreducible object if and only if M_λ is irreducible as a U_q-module.*

Proof. By definition, $\widehat{M_\lambda} = \widehat{U_q} \otimes_{\widehat{U_q^{\geq 0}}} L_\lambda$, and its topology is the quotient topology coming from $\widehat{U_q}$. Since it's therefore complete, it follows that $\widehat{M_\lambda} \cong \widehat{U_q} \widehat{\otimes}_{\widehat{U_q^{\geq 0}}} L_\lambda$. By Lemma

3.5.12 and using the fact that the projective tensor product is associative, we obtain an isomorphism

$$\widehat{M_\lambda} \cong \widehat{U_q^-} \widehat{\otimes}_L L_\lambda \cong \widehat{U_q^-} \otimes_L L_\lambda$$

as left $\widehat{U_q^-}$-modules. By considering now the $\widehat{U_q^0}$-action on this, and using the description of $\widehat{U_q^-}$ in (3.5), we see that $\widehat{M_\lambda} \in \hat{\mathcal{O}}$ and that $\widehat{M_\lambda}^{\mathrm{ss}} = U_q^- \otimes_L L_\lambda = M_\lambda$. The final two statements follow immediately from Lemma 3.5.10. □

Corollary 3.5.14. *Let $\lambda \in P$. Then the following are equivalent:*

- *$\widehat{M_\lambda}$ is an irreducible object in $\hat{\mathcal{O}}$.*
- *For every positive root β, $\langle \lambda + \rho, \beta^\vee \rangle \notin \mathbb{N}$.*

Proof. This is just the condition for M_λ to be irreducible, see [27, Corollary 10.1.11]. □

Definition 3.5.15. Given a coadmissible $\widehat{U_q}$-module $\mathcal{M}$ and $\lambda \in P$, an element $0 \neq m \in \mathcal{M}_\lambda$ is called a *maximal vector* of weight λ if $U_q^+ \cdot m = 0$. We say $\mathcal{M}$ is a *highest weight module with highest weight* λ if it is the cyclic $\widehat{U_q}$-module on a maximal vector in $\mathcal{M}_\lambda$.

From the proof of Proposition 3.5.13, we immediately obtain:

Lemma 3.5.16. *The coadmissible module $\widehat{M_\lambda}$ is a highest weight module with highest weight λ.*

Note more generally that it is immediate from the definition of $\hat{\mathcal{O}}$ that every object of $\hat{\mathcal{O}}$ contains a maximal vector. Hence by Proposition 3.5.9(iv), every irreducible object in $\hat{\mathcal{O}}$ is a highest weight module.

Proposition 3.5.17. *Let $\mathcal{M} \in \mathcal{C}(\widehat{U_q})$ be a highest weight module on a maximal vector $m \in \mathcal{M}$ of weight $\lambda \in P$. We have the following:*

(i) $\mathcal{M}$ is topologically $\widehat{U_q^0}$-semisimple with weights contained in $\lambda - Q^+$.

(ii) The weight spaces of $\mathcal{M}$ are finite dimensional and $\dim_L \mathcal{M}_\lambda = 1$. In particular, $\mathcal{M} \in \hat{\mathcal{O}}$ and $\mathcal{M}$ has finite length in $\hat{\mathcal{O}}$.

(iii) Each non-zero quotient of $\mathcal{M}$ by a coadmissible submodule is again a highest weight module.

(iv) Each coadmissible submodule of $\mathcal{M}$ generated by a maximal vector $m' \in \mathcal{M}_\mu$ for some $\mu < \lambda$ is proper. In particular, if $\mathcal{M}$ is an irreducible object in $\hat{\mathcal{O}}$ then all its maximal vectors lie in Lm, and hence $\mathrm{End}_{\widehat{U_q}}(\mathcal{M}) = L$.

(v) $\mathcal{M}$ has a unique maximal subobject and a unique irreducible quotient object and, hence, is an indecomposable object.

(vi) Let $\mathcal{N}$ be another highest weight module of weight μ. Then $\mathrm{End}_L \mathrm{Hom}_{\widehat{U_q}}(\mathcal{M}, \mathcal{N}) < \infty$. If $\lambda \neq \mu$ then $\mathcal{M}$ and $\mathcal{N}$ are not isomorphic. If $\mathcal{M}$ and $\mathcal{N}$ are simple objects and $\lambda = \mu$, then $\mathcal{M} \cong \mathcal{N}$.

Proof. By definition of highest weight modules, there is a surjection $\widehat{M_\lambda} \to \mathcal{M}$ which is a morphism in $\mathcal{C}(\widehat{U_q})$. Hence we see from Proposition 3.5.9(v) that $\mathcal{M} \in \hat{\mathcal{O}}$. From Corollary 3.5.6 and Proposition 3.5.13, we get a surjection

$$M_\lambda = \widehat{M_\lambda}^{\mathrm{ss}} \to \mathcal{M}^{\mathrm{ss}}.$$

In particular, $\mathcal{M}^{\mathrm{ss}}$ is a highest weight module of weight λ in $\mathcal{O}$. All properties therefore follow from the usual properties of $\mathcal{O}$ by Lemma 3.5.10. □

If we write $\widehat{L(\lambda)}$ to denote the unique irreducible quotient of $\widehat{M_\lambda}$, then we have $\widehat{L(\lambda)}^{\mathrm{ss}} \cong L(\lambda)$, where the latter denotes the unique irreducible quotient of M_λ, then we obtain:

Corollary 3.5.18. *The map $\lambda \mapsto [\widehat{L(\lambda)}]$ gives a bijection between P and the set of isomorphism classes of irreducible objects in $\hat{\mathcal{O}}$.*

We now describe a functor between the categories $\mathcal{O}$ and $\hat{\mathcal{O}}$. Recall from Remark 3.4.5 that there is an exact functor $F : M \mapsto \widehat{U_q} \otimes_{U_q} M$ between the category of finitely generated U_q-modules and the category of coadmissible $\widehat{U_q}$-modules. Moreover we have already seen that $F(M_\lambda) = \widehat{M_\lambda}$. Thus, if $M \in \mathcal{O}$ is a highest weight module of highest weight λ, then by exactness of F we get that $F(M)$ is a quotient of $\widehat{M_\lambda}$ and hence is in $\hat{\mathcal{O}}$. More generally, every object of $\mathcal{O}$ has a finite filtration with highest weight subquotients. Hence there is a surjection $\oplus_i M_i \to M$ from a finite direct sum of highest weight modules to M, and since F commutes with finite direct sums and is exact, it follows that $F(M)$ is a quotient of $\oplus_i F(M_i)$ and so lies in $\hat{\mathcal{O}}$. Hence F restricts to an exact functor

$$F : \mathcal{O} \to \hat{\mathcal{O}}.$$

Then we have:

Proposition 3.5.19. *The functor $F : \mathcal{O} \to \hat{\mathcal{O}}$ is a fully faithful exact embedding with left inverse given by $\mathcal{M} \mapsto \mathcal{M}^{\mathrm{ss}}$.*

Proof. It suffices to show that there is an isomorphism $M \cong F(M)^{\mathrm{ss}}$ natural in M. First observe that there is such a natural U_q-module map, given by $m \mapsto 1 \otimes m$. If $M = M_\lambda$ for some $\lambda \in P$, that map is an isomorphism by the proof of Proposition 3.5.13. If M is a highest weight module, we have a short exact sequence

$$0 \to N \to M_\lambda \to M \to 0$$

for some $\lambda \in P$. Writing N as a subquotient of U_q and using the fact that $\widehat{M_\lambda}$ is the completion of U_q/J_λ with the quotient locally convex topology, we see that the image of the map $F(N) \to \widehat{M_\lambda}$ is the closure of N in $\widehat{M_\lambda}$. Hence $N \cong F(N)^{\mathrm{ss}}$ by Proposition 3.5.7 and it follows that $M \cong F(M)^{\mathrm{ss}}$ by exactness of the two functors. Now if M is arbitrary, it has a filtration whose subquotients are highest weight modules. By induction we may assume M is an extension of highest weight modules. Then the result follows by the Five Lemma. □

Moreover we can easily identify the essential image of the functor F:

Lemma 3.5.20. *The essential image of F is the full subcategory of $\hat{\mathcal{O}}$ whose objects are those modules $\mathcal{M} \in \hat{\mathcal{O}}$ which have a finite filtration*

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_r = \mathcal{M}$$

by subobjects such that the quotient $\mathcal{M}_i/\mathcal{M}_{i-1}$ is a highest weight module for each $i \geq 1$.

Proof. The essential image is contained in this since, for $M \in \mathcal{O}$, we have an an analogous finite filtration in $\mathcal{O}$ with subquotients equal to highest weight modules and so we obtain the filtration for $F(M)$ by applying F to this filtration and using exactness. For the converse, suppose that $\mathcal{M}$ is as described. Then by exactness of $\mathcal{M} \mapsto \mathcal{M}^{\mathrm{ss}}$ (Corollary 3.5.6) and by Proposition 3.5.17 and its proof, we see that $\mathcal{M}^{\mathrm{ss}} \in \mathcal{O}$. Thus it suffices to show that $F(\mathcal{M}^{\mathrm{ss}}) \cong \mathcal{M}$. Now by applying the functor $\widehat{U_q} \otimes_{U_q} (\cdot)$ to the inclusion $\mathcal{M}^{\mathrm{ss}} \subset \mathcal{M}$ and postcomposing with the action map $u \otimes m \mapsto um$, we get a morphism $F(\mathcal{M}^{\mathrm{ss}}) \to \mathcal{M}$ in $\hat{\mathcal{O}}$. Let $\mathcal{K}$ and $\mathcal{C}$ denote its kernel and cokernel respectively. Then from Proposition 3.5.19 we get that $\mathcal{K}^{\mathrm{ss}} = \mathcal{C}^{\mathrm{ss}} = 0$, and so $\mathcal{K} = \mathcal{C} = 0$ by Proposition 3.5.7. $\square$

We claim that the full subcategory described in Lemma 3.5.20 is the whole of $\hat{\mathcal{O}}$, thus giving us a more precise version of Theorem B from the Introduction:

Theorem 3.5.21. *The functors F and $(\cdot)^{ss}$ are quasi-inverse equivalence of categories between the categories $\mathcal{O}$ and $\hat{\mathcal{O}}$.*

The above result was proved for (non-quantum) Arens-Michael envelopes in [72]. One of the main ingredients was a version of the Harish-Chandra isomorphism. Recall that the centre of $Z(U_q)$ is isomorphic to a polynomial algebra in n variables (see [49, Section 7.3, page 218] - note that this is only true for the simply connected form of the quantum group).

Conjecture 3.5.22. *The above isomorphism extends to a topological isomorphism $\widehat{Z(U_q)} \to \mathcal{O}(\mathbb{A}_L^{n,\mathrm{an}})$ between the closure of $Z(U_q)$ in $\widehat{U_q}$ and the algebra of rigid analytic functions on the analytification of affine n-space.*

To justify that this conjecture might plausibly be true, we show it for $U_q(\mathfrak{sl}_2)$. In that case, the centre $Z(U_q)$ is a polynomial algebra in the quantum Casimir element

$$C_q := FE + \frac{qK^2 + q^{-1}K^{-2}}{(q - q^{-1})^2},$$

see [48, Proposition 2.18]. In this $\mathfrak{sl}_2$ setting, recall that we had set the number m to be the least positive integer such that

$$\frac{\pi^{2m}}{q - q^{-1}} \in R.$$

Having recalled this, we can now show:

Proposition 3.5.23. *Conjecture 3.5.22 holds for $U_q(\mathfrak{sl}_2)$.*

Proof. By definition of C_q, for $n \geq 2m$, we have

$$\pi^{2n}C_q = (\pi^n F)(\pi^n E) + \frac{\pi^{2n}(qK^2 + q^{-1}K^{-2})}{(q-q^{-1})^2} \in U_n.$$

Hence we see that the subalgebra of $Z(U_q)$ consisting of polynomials in $\pi^{2n}C_q$ with coefficients in R is contained in the centre of U_n. Conversely, suppose that $z = \sum_{i=0}^{a} c_i C_q^i \in U_n$. We show by induction on a that each coefficient c_i belongs to $\pi^{2ni}R$. If $a = 0$ this is obvious so assume $a \geq 1$. Now note that

$$C_q^a = F^a E^a + (\text{terms of lower height})$$

with respect to the PBW basis for U_q. Indeed this follows from the commutator relation between E and F. Thus we see that the coefficient of $F^a E^a$ in the basis expression for z is c_a. But by the PBW theorem for U_n (Theorem 3.3.1) it follows that the coefficient of $F^a E^a$ in the basis expression for z is in $\pi^{2na}R$. Hence $c_a \in \pi^{2na}R$ and it follows that $c_a C_q^a \in R(\pi^{2n}C_q)^a \subseteq U_n$. Thus we may consider

$$\sum_{i=0}^{a-1} c_i C_q^i = z - c_a C_q^a \in U_n$$

and get that the other coefficients satisfy the required property by induction hypothesis.

The above calculation shows that the centre of U_n is $Z_n := R[\pi^{2n}C_q]$ for every $n \geq 2m$. If we write $\widehat{Z_{q,n}} := \widehat{Z_n} \otimes_R L$, we get that the closure $\widehat{Z(U_q)}$ of $Z(U_q)$ in $\widehat{U_q}$ is the projective limit $\varprojlim \widehat{Z_{q,n}}$. From our description of Z_n, it is clear that this is isomorphic to $\mathcal{O}(\mathbb{A}_L^{1,\mathrm{an}})$. $\square$

Remark 3.5.24. The non-quantum version of Harish-Chandra for the Arens-Michael envelope is due to Kohlhaase [53, Theorem 2.1.6]. The proof is relatively straightforward: essentially he shows that the same construction of the initial Harish-Chandra homomorphism extends to the Arens-Michael envelope. In our quantum setting, we can do that as well. One can straightforwardly construct a continuous projection map $\widehat{Z(U_q)} \to \widehat{U_q^0}$ and twist by $-\rho$, which gives a continuous algebra homomorphism with image in the Weyl group invariants. However all the defining norms of $\widehat{U_q}$ are identical on $\widehat{U_q^0}$ and so it is not clear a priori how to see the Fréchet structure of this image (this is something that does not occur in the classical situation).

The above calculation for $\mathfrak{sl}_2$ works because we have a complete and explicit description of the polynomial generator for the centre in terms of the PBW basis. In order to perform a similar calculation for a general Lie algebra, we'd need to have a similar description of the polynomial generators of the centre, something which we have not found in the literature.

We manage to prove Theorem 3.5.21 without this extended version of Harish-Chandra, simply by using the uncompleted Harish-Chandra instead. We first quickly recall some facts about central characters. For each $\lambda \in P$, the centre of U_q acts on the Verma module M_λ by a central character χ_λ (see [45, Lemma 6.3]). These characters satisfy the usual property that $\chi_\lambda = \chi_\mu$ if and only if $\mu \in W \cdot \lambda$ (see [27, Theorem 9.1.8]) with respect the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$. Thus every character has a unique representative in $-\rho + P^+$.

For a given $\lambda \in -\rho + P^+$, the character χ_λ extends to a continuous character of $\widehat{Z(U_q)}$, which we also denote by χ_λ, using the fact that $\mathrm{End}_{\hat{\mathcal{O}}}(\widehat{M_\lambda}) = L$ from Proposition 3.5.17(iv). Indeed it's clear from it that $\widehat{Z(U_q)}$ acts on the Verma module by a continuous character, and we see that this character extends χ_λ by considering the semisimple part. Hence we see more generally from Proposition 3.5.17 that $\widehat{Z(U_q)}$ acts on a highest weight module $\mathcal{M}$ by the character χ_λ, and that every Jordan-Holder factor of $\mathcal{M}$ must necessarily have highest weight in $W \cdot \lambda$.

Now, if $\mathcal{M} \in \hat{\mathcal{O}}$ then $Z(U_q)$ acts on each weight space $\mathcal{M}_\lambda$ and we may form the subspace

$$\mathcal{M}_\lambda^\chi := \{m \in \mathcal{M}_\lambda : (\ker \chi)^a \cdot m = 0 \text{ for some } a = a(m) \geq 1\}$$

where χ is a character of $Z(U_q)$. Since $\oplus_\lambda \mathcal{M}_\lambda^\chi$ is a U_q-submodule of $\mathcal{M}^{\mathrm{ss}}$, its closure $\mathcal{M}^\chi$ inside $\mathcal{M}$ is a subobject in $\hat{\mathcal{O}}$ by Lemma 3.5.10. Thus we may define the full subcategory $\hat{\mathcal{O}}^\chi$ of $\hat{\mathcal{O}}$ whose objects are those $\mathcal{M} \in \hat{\mathcal{O}}$ such that $\mathcal{M} = \mathcal{M}^\chi$. When $\chi = \chi_\mu$ for some $\mu \in P$, we write $\hat{\mathcal{O}}^\chi = \hat{\mathcal{O}}^\mu$. We now establish a few facts about these subcategories.

Lemma 3.5.25. *Suppose $\mathcal{M} \in \hat{\mathcal{O}}$ and χ is a central character as above. If $\mathcal{M}^\chi \neq 0$, then $\chi = \chi_\mu$ for some $\mu \in P$.*

Proof. Since $\mathcal{M}^\chi$ is an object in $\hat{\mathcal{O}}$, it must have a maximal vector $m \in \mathcal{M}_\mu^\chi$. Let $n \geq 1$ be minimal such that $(\ker \chi)^n \cdot m = 0$. Pick $0 \neq m' \in (\ker \chi)^{n-1} \cdot m$. Then m' is still a maximal vector and the centre acts on it by χ. On the other hand, the highest weight module generated by m' is a quotient of $\widehat{M_\mu}$ and hence the centre acts on it by χ_μ. This forces $\chi = \chi_\mu$. □

Hence we see that the only such subcategories which are non-zero are the $\hat{\mathcal{O}}^\mu$ for $\mu \in -\rho + P^+$.

Lemma 3.5.26. *The categories $\hat{\mathcal{O}}^\mu$ are abelian and the functor $\hat{\mathcal{O}} \to \hat{\mathcal{O}}^\mu$ given by $\mathcal{M} \mapsto \mathcal{M}^{\chi_\mu}$ is exact.*

Proof. Given a morphism $\mathcal{M} \to \mathcal{N}$ in $\hat{\mathcal{O}}$ we have morphisms $\mathcal{M}_\lambda \to \mathcal{N}_\lambda$ for each $\lambda \in P$ and $\mathcal{M}_\lambda^{\chi_\mu} \to \mathcal{N}_\lambda^{\chi_\mu}$. Taking the sum over all λ and passing to the closure, we get see that the assignement $\mathcal{M} \mapsto \mathcal{M}^{\chi_\mu}$ really is functorial. Since $\mathcal{M}_\lambda = \oplus_\chi \mathcal{M}_\lambda^\chi$, exactness of the map follows by the same argument using the fact that module maps between coadmissible modules are automatically strict and the passage to the closure then preserves exactness by Proposition 2.7.19(ii). As $\hat{\mathcal{O}}^\mu$ is a full subcategory of $\hat{\mathcal{O}}$, it is now clear that it is closed under passage to kernels and cokernels and, thus, abelian. □

Proposition 3.5.27. *The categories $\hat{\mathcal{O}}^\mu$ are Artinian and Noetherian.*

Proof. This follows using the classical argument for category $\mathcal{O}$ (see [44, Theorem 1.11]) as follows. Given $\mathcal{M} \in \hat{\mathcal{O}}^\mu$, let $V = \sum_{\lambda \in W \cdot \mu} \mathcal{M}_\lambda$. Then V is finite dimensional. Now if $0 \neq \mathcal{N}' \subset \mathcal{N}$ is a strict inclusion of subobjects of $\mathcal{M}$, let $m \in \mathcal{N}_\lambda$ be such that its image in $\mathcal{N}/\mathcal{N}'$ is a maximal vector for some weight λ. The cyclic submodule of $\mathcal{N}/\mathcal{N}'$ generated by the image of m is highest weight, hence $\widehat{Z(U_q)}$ acts on it by χ_λ. Hence it must be that $\chi_\lambda = \chi_\mu$ i.e. that $\lambda \in W \cdot \mu$. Thus by definition of V we see that $m \in \mathcal{N} \cap V$ and so we obtain $\dim_L(\mathcal{N} \cap V) > \dim_L(\mathcal{N}' \cap V)$. The result now follows. □

They key step in the proof of Theorem 3.5.21 is the following:

Proposition 3.5.28. *The above functors $\hat{\mathcal{O}} \to \hat{\mathcal{O}}^\mu$ induce a faithful embedding of $\hat{\mathcal{O}}$ into the direct product $\prod_{\mu \in -\rho + P^+} \hat{\mathcal{O}}^\mu$.*

Proof. Choose polynomial generators $z_1, \dots, z_n$ of $Z(U_q)$. Then for any $\mathcal{M} \in \hat{\mathcal{O}}$, the vector space $\mathcal{M}_\lambda^{\chi_\mu}$ is the simultaneous generalised eigenspace of the finitely many commuting operators $z_1, \dots, z_n$ with simultaneous generalised eigenvalues $\chi_\mu(z_1), \dots, \chi_\mu(z_n)$. Now there is a finite field extension $L \subseteq L'$ such that

$$\mathcal{M}_\lambda \otimes_L L' = \bigoplus_\chi (\mathcal{M}_\lambda \otimes_L L')^\chi$$

where the sum runs over a finite number of L'-valued characters of $Z(U_q)$ and $(\mathcal{M}_\lambda \otimes_L L')^\chi$ is defined in the obvious way. Hence we just need to show that if $(\mathcal{M}_\lambda \otimes_L L')^\chi \neq 0$ then $\chi = \chi_\mu$ for some μ. But this is Lemma 3.5.25, noting that $\mathcal{M} \otimes_L L'$ is in $\hat{\mathcal{O}}$ since L' is a finite extension.

Thus we have that $\mathcal{M}_\lambda = \bigoplus_\mu \mathcal{M}_\lambda^{\chi_\mu}$. Moreover, the equality $\mathcal{M}^\mu \cap \mathcal{M}^{\mathrm{ss}} = \bigoplus_\lambda \mathcal{M}_\lambda^{\chi_\mu}$ implies that $\mathcal{M}^{\mathrm{ss}} = \bigoplus_\mu (\mathcal{M}^\mu \cap \mathcal{M}^{\mathrm{ss}})$. Hence we see from this and the usual properties of $(\cdot)^{\mathrm{ss}}$ that the sum $\sum_\mu \mathcal{M}^\mu$ is direct and dense in $\mathcal{M}$. Indeed, to see that the sum is direct just note that, given any $\mu_1, \dots, \mu_n$, if

$$\mathcal{K} \to \bigoplus_{i=1}^n \mathcal{M}^{\mu_i} \to \mathcal{M}$$

is exact then we may apply $(\cdot)^{\mathrm{ss}}$ to obtain $\mathcal{K}^{\mathrm{ss}} = 0$ and so $\mathcal{K} = 0$. In particular the functor $\hat{\mathcal{O}} \to \prod_\mu \hat{\mathcal{O}}^\mu$ given by $\mathcal{M} \mapsto (\mathcal{M}^\mu)_\mu$ is faithful. □

We can now establish our main result. We first need a couple of preparatory results.

Lemma 3.5.29. *For every $n \geq m$, there is a triangular decomposition*

$$\widehat{U_{q,n}^-} \widehat{\otimes}_L \widehat{U_q^0} \widehat{\otimes}_L \widehat{U_{q,n}^+} \xrightarrow{\cong} \widehat{U_{q,n}}$$

given by the multiplication map.

Proof. By the PBW theorem (Theorem 3.3.1), the multiplication map yields a triangular decomposition

$$U_n^- \otimes_R U^0 \otimes_R U_n^+ \xrightarrow{\cong} U_n$$

for every $n \geq m$. The result now follows by Proposition 3.1.4. □

Given any coadmissible $\widehat{U_q}$-module $\mathcal{M}$, we write $\mathcal{M}_n := \widehat{U_{q,n}} \otimes_{\widehat{U_q}} \mathcal{M}$ which is a finitely generated Banach $\widehat{U_{q,n}}$-module. Moreover the canonical map $\mathcal{M} \to \mathcal{M}_n$ has dense image. We also have that the map $\widehat{U_q} \to \widehat{U_{q,n}}$ is flat for every $n \geq m$ (see [75, Remark 3.2]).

Lemma 3.5.30. *For any $\lambda \in P$ and any $n \geq m$, we have $\widehat{L(\lambda)}_n \neq 0$.*

Proof. Consider the kernel $\mathcal{K}$ of the surjection $\widehat{M_\lambda} \to \widehat{L(\lambda)}$. Since $\widehat{U_q} \to \widehat{U_{q,n}}$ is flat, the kernel of $(\widehat{M_\lambda})_n \to \widehat{L(\lambda)}_n$ is $\mathcal{K}_n$ for every $n \geq m$. By the triangular decomposition for $\widehat{U_{q,n}}$ from the previous Lemma, we get

$$(\widehat{M_\lambda})_n \cong \widehat{U_{q,n}} \otimes_{U_q} M_\lambda \cong \widehat{U_{q,n}^-} \otimes_L L_\lambda$$

and so $(\widehat{M_\lambda})_n$ is topologically $\widehat{U_q^0}$-semisimple with $((\widehat{M_\lambda})_n)^{\mathrm{ss}} = M_\lambda$. By Corollary 3.5.6, both $\mathcal{K}_n$ and $\widehat{L(\lambda)}_n$ are topologically semisimple and it suffices to show that $\mathcal{K}_n^{\mathrm{ss}} \neq ((\widehat{M_\lambda})_n)^{\mathrm{ss}} = M_\lambda$. Now the composite $\mathcal{K}^{\mathrm{ss}} \subset \mathcal{K} \to \mathcal{K}_n$ has dense image, so it follows from Proposition 3.5.7 that its image is $\mathcal{K}_n^{\mathrm{ss}}$. So we get $\mathcal{K}_n^{\mathrm{ss}} \cong \mathcal{K}^{\mathrm{ss}}$ as U_q^0-modules, and now we see that $\mathcal{K}_n^{\mathrm{ss}} \neq M_\lambda$ as required because $\widehat{L(\lambda)}^{\mathrm{ss}} \neq 0$. □

Proposition 3.5.31. *The category $\hat{\mathcal{O}}$ is Artinian and Noetherian.*

Proof. Let $\mathcal{M} \in \hat{\mathcal{O}}$. We have from the proof of Proposition 3.5.28 that $\bigoplus_\mu \mathcal{M}^\mu$ is dense in $\mathcal{M}$. Now for any $n \geq m$, we have

$$\mathcal{M}_n = \widehat{U_{q,n}} \otimes_{\widehat{U_q}} \mathcal{M} \supseteq \widehat{U_{q,n}} \otimes_{\widehat{U_q}} \left(\bigoplus_\mu \mathcal{M}^\mu \right) = \bigoplus_\mu (\mathcal{M}^\mu)_n.$$

Any non-zero $\mathcal{M}^\mu$ has a composition series by Proposition 3.5.27 and so $\widehat{L(\lambda)}_n \subseteq (\mathcal{M}^\mu)_n$ for some $\lambda \in P$ and then we see that $(\mathcal{M}^\mu)_n \neq 0$ by the previous Lemma. Since $\mathcal{M}_n$ is a finitely generated $\widehat{U_{q,n}}$-module and $\widehat{U_{q,n}}$ is Noetherian, it follows that $\mathcal{M}^\mu = 0$ for all but finitely many μ. But then the sum $\bigoplus_\mu \mathcal{M}^\mu$ is finite and so closed by Proposition 3.5.9(iii)&(v). □

Proof of Theorem 3.5.21. This follows immediatly from the previous Proposition by Lemma 3.5.20. □

Chapter 4

Quantum flag varieties and their integral forms

We now recall the construction of the quantum flag variety from [13] and make analogous constructions for integral forms. Throughout, we will often use without mention Theorem 2.5.12 and Remark 2.5.13, i.e. the fact that comodules over $\mathcal{O}_q$, $\mathcal{A}_q$, $\mathcal{O}_q(B)$ and $\mathcal{B}_q$ are the same thing as integrable modules over U_q, U^{res}, $U_q^{\geq 0}$ and $U^{\mathrm{res}}(\mathfrak{b})$ respectively.

4.1 Recap on the quantum flag variety

In this Section we review definitions and results from [13] which we shall adapt or use later. We first begin by recalling the definition of the quantum flag variety. Since $\mathcal{O}_q(B)$ is a quotient Hopf algebra of $\mathcal{O}_q$, the comultiplication on $\mathcal{O}_q$ induces a map

$$\mathcal{O}_q \to \mathcal{O}_q \otimes_L \mathcal{O}_q \to \mathcal{O}_q \otimes_L \mathcal{O}_q(B)$$

which by abuse of notation we will also denote by Δ. This naturally makes $\mathcal{O}_q$ into an $\mathcal{O}_q(B)$-comodule. In the integrable modules language, this is simply the restriction to $U_q^{\geq 0}$ of the U_q-module structure on $\mathcal{O}_q$.

Definition 4.1.1. ([13, Definition 3.1]) A *B_q-equivariant sheaf on G_q* is a triple (F, α, β) where F is an L-vector space, $\alpha : \mathcal{O}_q \otimes F \to F$ is a left $\mathcal{O}_q$-module action and $\beta : F \to F \otimes \mathcal{O}_q(B)$ is a right $\mathcal{O}_q(B)$-comodule action, such that α is a comodule homomorphism where $\mathcal{O}_q \otimes F$ is given the tensor comodule structure. We denote by $\mathcal{M}_{B_q}(G_q)$ the category of B_q-equivariant sheaves on G_q.

Remark 4.1.2. In the classical case $q = 1$, this category is equivalent to the category of B-equivariant sheaves of $\mathcal{O}_G$-modules, which in turn is equivalent to the category of quasi-coherent sheaves of $\mathcal{O}_{G/B}$-modules. So the category $\mathcal{M}_{B_q}(G_q)$ can be thought of as the quantum analogue of the flag variety.

By the above, $\mathcal{O}_q$ is an object of this category. More generally we have a notion of line bundles. Any element $\lambda \in T_P$ may be thought of as a character of the group algebra $LP \cong L[K_\mu : \mu \in P]$, and we may extend it to a character of $U_q^{\geq 0}$ by setting it to kill the

E's. This defines a one dimensional $U_q^{\geq 0}$-module L_λ. The ones among these which are integrable, and so $\mathcal{O}_q(B)$-comodules, correspond to $\lambda \in P$, and the coaction is $1 \mapsto 1 \otimes \lambda$.

Definition 4.1.3. ([13, Definition 3.3]) We define a line bundle in $\mathcal{M}_{B_q}(G_q)$ to be an object of the form $\mathcal{O}_q(\lambda) := \mathcal{O}_q \otimes_L L_{-\lambda}$ for $\lambda \in P$, where the $\mathcal{O}_q$-action is on the left factor and the $\mathcal{O}_q(B)$-coaction is the tensor one. More generally for a finite dimensional $\mathcal{O}_q(B)$-comodule V we get that $\mathcal{O}_q \otimes_L V$ with an analogous structure as above is an element of $\mathcal{M}_{B_q}(G_q)$ and we may think of it as a vector bundle.

Now that we have a flag variety, we turn to the notion of taking global sections.

Definition 4.1.4. ([13, Definition 3.4]) The *global section functor* $\Gamma : \mathcal{M}_{B_q}(G_q) \to L$-v.s is defined to be

$$\Gamma(M) := \mathrm{Hom}_{\mathcal{M}_{B_q}(G_q)}(\mathcal{O}_q, M) = \{m \in M : \beta(m) = m \otimes 1\} =: M^{B_q},$$

which we call the B_q-*invariants* of M.

Next, we want to discuss the analogue of sheaf cohomology in this context. In order to do that, we have:

Lemma 4.1.5. ([13, Lemma 3.8]) *The categories $\mathcal{M}_{B_q}(G_q)$ and $\mathcal{O}_q(B)$-comod have enough injectives.*

Hence we can right derive the global section functor. It was shown in [13, Section 3] that the category $\mathcal{M}_{B_q}(G_q)$ is equivalent to a Proj category in the sense of Artin-Zhang [9]. That includes [13, Proposition 3.5] which states that the line bundles are very ample in the sense that for any coherent module M, the twist $M(\lambda)$ is Γ-acyclic and generated by its global sections for $\lambda >> 0$. As we will reprove these for integral forms, we postpone the definitions until then.

We next turn to D-modules. There is a left U_q-module algebra structure on $\mathcal{O}_q$ given by

$$u(a) = \sum a_2(u) \cdot a_1, \tag{4.1}$$

for $u \in U_q$ and $a \in \mathcal{O}_q$. By viewing $\mathcal{O}_q \subseteq U_q^*$, this action amounts to the action $u(a)(x) = a(xu)$ for $a \in \mathcal{O}_q$ and $u, x \in U_q$. Following [13, Definition 4.1], we define the ring of quantum differential operators on G_q to be the smash product algebra $\mathcal{D}_q = \mathcal{O}_q \# U_q$. This is a natural choice by Example 2.1.18.

Recall that in the smash product algebra $\mathcal{D}_q$, for all $u \in U_q$ and all $a \in \mathcal{O}_q$, the adjoint action $\sum u_1 a S(u_2)$ inside $\mathcal{D}_q$ coincides with the above action $u(a)$. We will need the following result which was not proved in [13]:

Proposition 4.1.6. *The ring $\mathcal{D}_q$ is Noetherian.*

Proof. Since $\mathcal{D}_q = \mathcal{O}_q \otimes_L U_q$ as a vector space, and since $x \cdot (yu) = (xy)u$ for all $x, y \in \mathcal{O}_q$ and all $u \in U_q$, it follows that $\mathcal{D}_q$ is generated as an $\mathcal{O}_q$-module by U_q. Recall our height filtration on U_q. We now define an analogous filtration on $\mathcal{D}_q$ given by

$$F_i\mathcal{D}_q = \mathcal{O}_q \cdot F_iU_q.$$

We claim this defines an algebra filtration. Indeed, suppose that for some $i, j \geq 0$, we are given $u \in F_i U_q$ and $v \in F_j U_q$, and take $x, y \in \mathcal{O}_q$. By definition of the Hopf algebra structure on U_q, we have that $\Delta(u) \in F_i(U_q \otimes_L U_q) \subset F_i U_q \otimes_L F_i U_q$ where we give $U_q \otimes_L U_q$ the tensor filtration. Therefore, it follows that

$$(xu)(yv) = \sum \big(xu_1(y)\big)\big(u_2 v\big) \in F_{i+j}\mathcal{D}_q$$

since the filtration on U_q is an algebra filtration. Hence $\mathcal{D}_q$ is a positively filtered L-algebra.

It will therefore be enough to show that $\operatorname{gr}\mathcal{D}_q$ is Noetherian. First, observe that $F_0\mathcal{D}_q$ is generated over $\mathcal{O}_q$ by the K_μ for $\mu \in P$, which all commute. Moreover, for each generator x_i of $\mathcal{O}_q$, we have that

$$K_\mu x_i K_{-\mu} = K_\mu(x_i) \in q^{\frac{1}{d}\mathbb{Z}} x_i$$

by definition of the U_q-action on $\mathcal{O}_q$ and since the x_i's are matrix coefficients with respect to weight bases. Thus we see that the generators of $F_0 U_q$ normalise $\mathcal{O}_q$. Hence it follows that $F_0\mathcal{D}_q$ is Noetherian by Lemma 2.5.6 since $\mathcal{O}_q$ is Noetherian by Theorem 2.5.4.

Next, we claim that the symbols $\overline{E_{\alpha_i}}$ and $\overline{F_{\alpha_j}}$ normalise $F_0\mathcal{D}_q$ in $\operatorname{gr}\mathcal{D}_q$ for all i, j. Indeed, we have that they q-commute with the K's and for $x \in \mathcal{O}_q$, we have

$$E_{\alpha_i} x - (K_{\alpha_i} x K_{-\alpha_i}) E_{\alpha_i} = E_{\alpha_i}(x) \in \mathcal{O}_q \subseteq F_0\mathcal{D}_q,$$

where $K_{\alpha_i} x K_{-\alpha_i} \in \mathcal{O}_q$ by the above. Thus in $\operatorname{gr}\mathcal{D}_q$ we have

$$\overline{E_{\alpha_i}} x = (K_{\alpha_i} x K_{-\alpha_i}) \overline{E_{\alpha_i}} \in F_0\mathcal{D}_q \cdot \overline{E_{\alpha_i}}.$$

Similarly for the F's.

Finally we give to $\operatorname{gr}\mathcal{D}_q$ an analogue of the $\mathbb{Z}^{2N}_{\geq 0}$-filtration on $\operatorname{gr} U_q$ from Theorem 2.4.10. More precisely, we make $\operatorname{gr}\mathcal{D}_q$ into a $\mathbb{Z}^{2N}_{\geq 0}$-filtered $F_0\mathcal{D}_q$-algebra. First we impose the reverse lexicographic total ordering on $\mathbb{Z}^{2N}_{\geq 0}$, and give a $\mathbb{Z}^{2N}_{\geq 0}$-filtration on $\operatorname{gr}\mathcal{D}_q$ by stating that a monomial

$$F_{\beta_1}^{r_1} \cdots F_{\beta_N}^{r_N} K_\lambda E_{\beta_1}^{s_1} \cdots E_{\beta_N}^{s_N}$$

has degree $(r_1, \ldots, r_N, s_1, \ldots, s_N)$. Then it follows from Theorem 2.4.10 that the corresponding associated multigraded algebra is a q-commutative $F_0\mathcal{D}$-algebra. Hence the associated graded algebra of $\operatorname{gr}\mathcal{D}_q$ is Noetherian, and so it must be that $\operatorname{gr}\mathcal{D}_q$ is Noetherian. $\square$

Note that $\mathcal{D}_q$ is a U_q-module algebra via the adjoint action in $\mathcal{D}_q$, or alternatively by tensoring the U_q-action (4.1) on $\mathcal{O}_q$ with the adjoint action on U_q. Explicitly,

$$u \cdot (a \otimes v) = \sum u_1(a) \otimes u_2 v S(u_3). \tag{4.2}$$

We now are ready to define D-modules on the quantum flag variety:

Definition 4.1.7. ([13, Definition 4.2]) Let $\lambda \in T_P$. A *(B_q, λ)-equivariant $\mathcal{D}_q$-module* is a triple (M, α, β) where M is an L-vector space, $\alpha : \mathcal{D}_q \otimes M \to M$ is a left $\mathcal{D}_q$-module action and $\beta : M \to M \otimes \mathcal{O}_q(B)$ is a right $\mathcal{O}_q(B)$-comodule action. The map β induces a left $U_q^{\geq 0}$-action on M which we also denote by β. These actions must satisfy:

(i) The $U_q^{\geq 0}$-actions on $M \otimes L_\lambda$ given by $\beta \otimes \lambda$ and $\alpha|_{U_q^{\geq 0}} \otimes 1$ are equal.

(ii) The map α is $U_q^{\geq 0}$-linear with respect to the β-action on M and the action (4.2) on $\mathcal{D}_q$.

In other words M is an object of $\mathcal{M}_{B_q}(G_q)$ equipped with a $U_q^{\geq 0}$-equivariant $\mathcal{D}_q$-action with in addition the condition (i).

We denote by $\mathcal{D}^\lambda_{B_q}(G_q)$ the category of such $\mathcal{D}_q$-modules. We have a forgetful functor $\mathcal{D}^\lambda_{B_q}(G_q) \to \mathcal{M}_{B_q}(G_q)$, which allows us to define a global section functor on $\mathcal{D}^\lambda_{B_q}(G_q)$ given by $\Gamma \circ$ forget. We also denote this functor by Γ.

Note that condition (i) above can be rephrased into saying that for $M \in \mathcal{D}^\lambda_{B_q}(G_q)$ and $m \in M$, we have $E_\alpha m = \beta(E_\alpha)m$ and $K_\mu m = \lambda(\mu)\beta(K_\mu)m$ for all simple roots α and $\mu \in P$. In particular if m is a global section then by B_q-invariance we must have $E_\alpha m = 0$ and $K_\mu m = \lambda(\mu)m$. In other words global sections consist of the highest weight vectors of weight λ. So we see that the $\mathcal{D}_q$-module homomorphisms $\mathcal{D}_q \to M$ corresponding to global sections kill the left ideal $\mathcal{D}_q I$ where $I = \{E_{\alpha_i}, K_\mu - \lambda(K_\mu) : 1 \leq i \leq n, \mu \in P\}$.

Based on the above, we define $\mathcal{D}^\lambda_q$ to be the quotient

$$\mathcal{D}^\lambda_q = \mathcal{D}_q / \mathcal{D}_q I$$

where I is as above. We can see that $\mathcal{D}^\lambda_q = \mathcal{O}_q \otimes_L M_\lambda$ where M_λ is the Verma module of highest weight λ. Recall from Lemma 2.4.17 that there is a surjection $U_q^{\text{fin}} \to M_\lambda$. Using this, we can view M_λ as an $\mathcal{O}_q(B)$-comodule, or an integrable $U_q^{\geq 0}$-module, via the quotient of the adjoint action. This action is just the usual action twisted by $-\lambda$ and so with this $U_q^{\geq 0}$-module structure it is isomorphic to $M_\lambda \otimes L_{-\lambda}$ and has trivial highest weight. Then, as an object of $\mathcal{M}_{B_q}(G_q)$, $\mathcal{D}^\lambda_q = \mathcal{O}_q \otimes_L M_\lambda$ with the tensor $\mathcal{O}_q(B)$-coaction and with the action of $\mathcal{O}_q$ on the left factor, where we view M_λ is an $\mathcal{O}_q(B)$-comodule just as now. It's moreover in $\mathcal{D}^\lambda_{B_q}(G_q)$: (i) follows from our discussion above of the fact that M_λ has trivial highest weight as an $\mathcal{O}_q(B)$-comodule, and (ii) simply follows from the fact that $\mathcal{D}_q$ is a $U_q^{\geq 0}$-module algebra.

Then $\mathcal{D}^\lambda_q$ represents the global section functor, i.e. $\Gamma(M) = \mathrm{Hom}_{\mathcal{D}^\lambda_{B_q}(G_q)}(\mathcal{D}^\lambda_q, M)$ by the above. In particular, $\Gamma(\mathcal{D}^\lambda_q)$ is a ring. Also one can easily check that $\mathcal{D}^\lambda_q$ is the maximal quotient of $\mathcal{D}_q$ that lies in $\mathcal{D}^\lambda_{B_q}(G_q)$, where we take the quotient $\mathcal{D}_q$-action and the quotient of the $U_q^{\geq 0}$-action (4.2) on $\mathcal{D}_q$.

Definition 4.1.8. Let M_λ be the Verma module with highest weight λ. Let $J_\lambda = \mathrm{Ann}_{U_q^{\text{fin}}}(M_\lambda)$. We write $U^\lambda_q = U_q^{\text{fin}} / J_\lambda$.

We finally recall the notion of regular and dominant weights in this context. Recall from the discussion following Remark 3.5.24 that the centre $Z(U_q)$ of U_q acts on any Verma module M_λ by a character χ_λ. Following [13, 2.1], we say that $\lambda \in T_P$ is *dominant* if $\chi_\lambda \neq \chi_{\lambda+\mu}$ for any $0 \neq \mu \in Q^+$, and that λ is regular dominant if for all $\mu \in P^+$ and all weight $\gamma \neq \mu$ of $V(\mu)$, we have $\chi_{\lambda+\mu} \neq \chi_{\lambda+\gamma}$. When $\lambda \in P$ this is equivalent to saying that it's dominant, respectively regular dominant in the classical sense.

Theorem 4.1.9 ([13, Proposition 4.8 and Theorem 4.12]). *Suppose that $\lambda \in T_P$ is regular and dominant. Then*

$$U_q^\lambda \cong \Gamma(\mathcal{D}_q^\lambda)^{op}$$

and we have an equivalence of categories

$$\Gamma : \mathcal{D}_{B_q}^\lambda(G_q) \to \Gamma(\mathcal{D}_q^\lambda)\text{-}mod.$$

whose inverse is given by the localisation functor $\mathrm{Loc}(M) = \mathcal{D}_q^\lambda \otimes_{\Gamma(\mathcal{D}_q^\lambda)} M$.

4.2 The integral quantum flag variety and proj categories

In this Section, we investigate integral forms of the categories from the previous Section, and prove that it is a noncommutative projective scheme. A lot of our arguments are similar to those of [13, Section 3].

We first return to our integral forms $\mathcal{A}_q$ and $\mathcal{B}_q$ and make completely analogous definitions to the previous section. Again, since $\mathcal{B}_q$ is a quotient Hopf algebra of $\mathcal{A}_q$, the comultiplication on $\mathcal{A}_q$ induces a map

$$\mathcal{A}_q \to \mathcal{A}_q \otimes_R \mathcal{A}_q \to \mathcal{A}_q \otimes_R \mathcal{B}_q$$

which makes $\mathcal{A}_q$ into a $\mathcal{B}_q$-comodule. By abuse of notation, we will also denote this map by Δ.

Definition 4.2.1. The integral quantum flag variety is the category $\mathscr{C}_R$ whose objects consist of $\mathcal{A}_q$-modules M which are equipped with a right $\mathcal{B}_q$-comodule structure $M \to M \otimes_R \mathcal{B}_q$ such that the $\mathcal{A}_q$-action map $\mathcal{A}_q \otimes_R M \to M$ is a comodule homomorphism where we give $\mathcal{A}_q \otimes_R M$ the tensor comodule structure. The morphisms are just the $\mathcal{A}_q$-linear maps which are also comodule homomorphisms.

There is an obvious functor

$$\begin{aligned} \mathscr{C}_R &\longrightarrow \mathcal{M}_{B_q}(G_q) \\ M &\longmapsto M_L := M \otimes_R L \end{aligned}$$

to our quantum flag variety. Given $M \in \mathscr{C}_R$, we will write ρ_M (respectively ρ_{M_L}) to denote the comodule map on M (respectively M_L).

Next, there are several adjunctions we need to describe. Namely we have

$$\begin{array}{ccc} \mathcal{A}_q\text{-mod} & \xrightarrow{q} & R\text{-mod} \\ \Big\downarrow{\scriptstyle\theta} & & \Big\downarrow{\scriptstyle\phi} \\ \mathscr{C}_R & \xrightarrow{p} & \mathcal{B}_q\text{-comod} \end{array}$$

where each arrow denotes a pair of functors. We write (θ^*, θ_*), (p^*, p_*), (q^*, q_*) and (ϕ^*, ϕ_*) where each time the 'lower star' functors are the right adjoints and go in the direction of the arrows. The functor $\theta_* : \mathcal{A}_q\text{-mod} \to \mathscr{C}_R$ is given by $N \mapsto N \otimes_R \mathcal{B}_q$ where $\mathcal{A}_q$ acts on $\theta_*(N)$ via the tensor action and the $\mathcal{B}_q$-coaction comes from the second factor, while $\theta^* : \mathscr{C}_R \to \mathcal{A}_q$-mod is just the forgetful functor. The bijection making this an adjunction is given as follows. Let $M \in \mathscr{C}_R$ and $N \in \mathcal{A}_q$-mod, and let $\rho : M \to M \otimes_R \mathcal{B}_q$ and $\varepsilon : \mathcal{B}_q \to R$ be the comodule map and the counit of $\mathcal{B}_q$ respectively. Given a module homomorphism $f : M \to N$, we construct a morphism $g : M \to N \otimes_R \mathcal{B}_q$ in $\mathscr{C}_R$ by taking the composite $(f \otimes \mathrm{id}) \circ \rho$. Conversely, given a morphism $g : M \to N \otimes_R \mathcal{B}_q$ in $\mathscr{C}_R$, we construct a module homomorphism $f : M \to N$ by taking the composite $(\mathrm{id} \otimes \varepsilon) \circ g$.

Moreover the adjunction between $\mathscr{C}_R$ and $\mathcal{B}_q$-comod is given by $p_* =$ forgetful one way and the functor $p^* : M \mapsto \mathcal{A}_q \otimes_R M$ the other way, where $\mathcal{A}_q$ acts on the first factor and the $\mathcal{B}_q$-coaction is the tensor coaction. The bijection is as follows: given a map $f : \mathcal{A}_q \otimes_R M \to N$ in $\mathscr{C}_R$ we get a comodule map $M \to N$ by taking $m \mapsto f(1 \otimes m)$, and conversely given a comodule map $g : M \to N$ we get a map $\mathcal{A}_q \otimes_R M \to N$ by post-composing $\mathrm{id} \otimes g : \mathcal{A}_q \otimes_R M \to \mathcal{A}_q \otimes_R N$ with the action map $\mathcal{A}_q \otimes_R N \to N$.

Similarly $q_* =$ forgetful, $q^* : M \mapsto \mathcal{A}_q \otimes_R M$, $\phi^* =$ forgetful and $\phi_* : M \to M \otimes_R \mathcal{B}_q$ where the coaction is on the second factor, all with similar bijections as in the above.

In particular, the maps $M \to \theta_*\theta^*(M)$ and $M \to \phi_*\phi^*(M)$ are both just the comodule map ρ and so are injective, since ρ has left inverse $1 \otimes \varepsilon$. Also note that since $\mathcal{A}_q$ and $\mathcal{B}_q$ are torsion-free and so flat over R, all the functors are exact and so θ_*, p_*, q_* and ϕ_* all map injective objects to injective objects by Lemma 2.9.8.

Lemma 4.2.2. *The categories $\mathscr{C}_R$ and $\mathcal{B}_q$-comod have enough injectives.*

Proof. Let $M \in \mathscr{C}_R$ and let I be an injective $\mathcal{A}_q$-module such that there is an $\mathcal{A}_q$-linear injection $M \to I$. By the above, the adjunction map $M \to \theta_*\theta^*(M)$ is injective, and so there is an injection

$$M \to \theta_*\theta^*(M) \to \theta_*(I).$$

But by the above θ_* preserves injectives, so $\theta_*(I) = I \otimes_R \mathcal{B}_q$ is injective and we're done for $\mathscr{C}_R$. The proof for $\mathcal{B}_q$-comod is entirely analogous working with ϕ instead of θ. $\square$

Definition 4.2.3. We define the global sections functor $\Gamma : \mathscr{C}_R \to R$-mod to be

$$\Gamma(M) := \mathrm{Hom}_{\mathscr{C}_R}(\mathcal{A}_q, M) = \{m \in M : \rho(m) = m \otimes 1\} := M^{\mathcal{B}_q}.$$

Note that the above lemma implies that we can right derive this functor.

Our first main aim is to show that our category $\mathscr{C}_R$ is a noncommutative projective scheme in the sense of Artin-Zhang [9, 2.3-2.4]. We quickly recall the definitions. Given a $\mathbb{Z}^n$-graded ring $\mathcal{R} = \oplus_{\mathbf{m} \in \mathbb{Z}^n} \mathcal{R}_{\mathbf{m}}$, we say that a graded (left or right) $\mathcal{R}$-module M is *torsion* if, for every $m \in M$, there exists some a such that m is killed by $\mathcal{R}_{\geq a} := \oplus_{m_1, \ldots, m_n \geq a} \mathcal{R}_{\mathbf{m}}$. Write $\mathcal{R}$-mod to denote the category of *graded* (left or right) $\mathcal{R}$-modules. The full subcategory $\mathcal{T}(\mathcal{R})$ of torsion modules is a Serre subcategory of $\mathcal{R}$-mod, and we let $\mathrm{Proj}(\mathcal{R}) := \mathcal{R}\text{-mod}/\mathcal{T}(\mathcal{R})$ denote the quotient category. Similarly, when $\mathcal{R}$ is graded (left or right) Noetherian, we denote by $\mathrm{proj}(\mathcal{R})$ the quotient category of the category of finitely generated graded modules by the full subcategory of finitely generated torsion modules.

Now suppose that we are equipped with a tuple $(\mathcal{C}, \mathcal{O}, s_1, \ldots, s_n)$ where $\mathcal{C}$ is an abelian category, $\mathcal{O}$ is an object of $\mathcal{C}$ and $s_1, \ldots, s_n$ are pairwise commuting autoequivalences of $\mathcal{C}$. For $\mathbf{m} \in \mathbb{Z}^n$ and an object M of $\mathcal{C}$, we define twisting functors on $\mathcal{C}$ by

$$M(\mathbf{m}) = s_1^{m_1} \cdots s_n^{m_n}(M).$$

We let Γ denote the functor $\mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, -)$ and we set $\underline{\Gamma}(M) = \oplus_{\mathbf{m} \in \mathbb{N}^n} \Gamma(M(\mathbf{m}))$. Note that $\underline{\Gamma}(\mathcal{O})$ is a graded ring where the multiplication is defined as follows: for $a \in \Gamma(\mathcal{O}(\mathbf{m}))$ and $b \in \Gamma(\mathcal{O}(\mathbf{m}'))$, we set

$$a \cdot b := s_1^{m_1'} \cdots s_n^{m_n'}(a) \circ b.$$

Similarly for each M in $\mathcal{C}$, $\underline{\Gamma}(M)$ is a graded right $\underline{\Gamma}(\mathcal{O})$-module. Finally, let $\mathcal{C}^0$ denote the full subcategory of Noetherian objects in $\mathcal{C}$. Then we have the following multigraded version of a result of Artin and Zhang (see also [13, Proposition 2.1]):

Proposition 4.2.4 ([9, Theorem 4.5], [13, Remark 2.2]). *Let $(\mathcal{C}, \mathcal{O}, s_1, \ldots, s_n)$ be a tuple as above, such that the following hold:*

(i) $\mathcal{O}$ belongs to $\mathcal{C}^0$;

(ii) $\Gamma(\mathcal{O})$ is a right Noetherian ring and $\Gamma(M)$ is a finitely generated $\Gamma(\mathcal{O})$-module for each object M of $\mathcal{C}^0$;

(iii) for each $M \in \mathcal{C}^0$ there is an epimorphism $\oplus_{i=1}^{l} \mathcal{O}(-\mathbf{m_i}) \to M$ for some $l \geq 1$ and $\mathbf{m_1}, \ldots, \mathbf{m_l} \in \mathbb{N}^n$; and

(iv) given $M, N \in \mathcal{C}^0$ and an epimorphism $M \to N$ in $\mathcal{C}$, the associated map $\Gamma(M(\mathbf{m})) \to \Gamma(N(\mathbf{m}))$ is surjective for $\mathbf{m} >> 0$.

Then $\underline{\Gamma}(\mathcal{O})$ is right Noetherian and $\mathcal{C}^0$ is equivalent to $\mathrm{proj}(\underline{\Gamma}(\mathcal{O}))$ (working with graded right modules). If, moreover, we assume that every object of $\mathcal{C}$ is a direct limit of objects in $\mathcal{C}^0$, then $\mathcal{C}$ is equivalent to $\mathrm{Proj}(\underline{\Gamma}(\mathcal{O}))$.

Note that in general the assignement $M \mapsto \underline{\Gamma}(M)$ defines a left exact functor from $\mathcal{C}$ to the category of graded $\underline{\Gamma}(\mathcal{O})$-modules. Now we return to the setting of the quantum R-flag variety.

Definition 4.2.5. We define the *representation ring* to be $R_q := \oplus_{\lambda \in P^+} \Gamma(\mathcal{A}_q(\lambda))$ with the induced ring structure from the multiplication in $\mathcal{A}_q$.

Remark 4.2.6. We will apply the above setup to the category $\mathscr{C}_R$. Specifically we will set the autoequivalences to be $s_i(M) := M(\varpi_i)$. The above mentioned ring structure for $\underline{\Gamma}(\mathcal{A}_q)$ is then just the ring structure of R_q^{op}. So we will apply the above results, working with R_q, by replacing every instance of the word 'right' by 'left'.

Theorem 4.2.7. *The category $\mathscr{C}_R$ is equivalent to* $\mathrm{Proj}(R_q)$ *(this time working with left modules).*

We now start preparing for the proof of this theorem. We begin by proving results in $\mathscr{C}_R$ analogous to standard facts about line bundles on the flag variety. We will mostly just adapt arguments from [13, Section 3.4]. They apply essentially identically but we repeat them nevertheless.

Note that we have a functor of taking $\mathcal{B}_q$-invariants in $\mathcal{B}_q$-comod, which we denote by $\tilde{\Gamma}$. This functor is also left exact. Note that the functor $\Gamma \circ p^* : M \mapsto (\mathcal{A}_q \otimes_R M)^{\mathcal{B}_q}$ is just the induction functor $\mathrm{Ind} : M \mapsto (\mathcal{A}_q \otimes_R M)^{U^{\mathrm{res}}(\mathfrak{b})}$ from Definition 2.5.14. This will be useful in the next result.

Proposition 4.2.8. *(i) If $I \in \mathcal{B}_q$-comod is injective, then $p^*(I)$ is Γ-acyclic.*

(ii) For any $M \in \mathcal{B}_q$-comod and any $i \geq 0$, $R^i\mathrm{Ind}(M) = R^i\Gamma(p^*(M))$.

(iii) For any $M \in \mathscr{C}_R$ and any $i \geq 0$, $R^i\tilde{\Gamma}(p_*(M)) \cong R^i\Gamma(M)$.

(iv) The functor Γ has cohomological dimension at most $N = \dim G/B$.

Proof. (i) The adjunction map $I \to \phi_*\phi^*(I) = I \otimes_R \mathcal{B}_q$ is injective. So as I is injective, this embedding splits. Therefore, as p^* is additive and since the derived functors $R^i\Gamma$ commute with finite direct sums, it suffices to show that $p^*(I \otimes_R \mathcal{B}_q)$ is acyclic. To simplify notation a bit, we write $J = \phi^*(I)$. We claim that we have an isomorphism $p^*(\phi_*(J)) \xrightarrow{\cong} \theta_*(q^*(J))$. Indeed, as R-modules they both equal $\mathcal{A}_q \otimes_R I \otimes_R \mathcal{B}_q$ and the isomorphism is given by $a \otimes i \otimes b \mapsto \sum a_1 \otimes i \otimes a_2 b$, with inverse $a \otimes i \otimes b \mapsto \sum a_1 \otimes i \otimes S(a_2)b$. These maps are easily checked to be both module and comodule homomorphisms. Hence we have that

$$\begin{aligned} R^i\Gamma(p^*(I \otimes_R \mathcal{B}_q)) &\cong R^i\Gamma(\theta_*(q^*(J))) \\ &= \mathrm{Ext}^i_{\mathscr{C}_R}(\mathcal{A}_q, \theta_*(q^*(J))) \\ &\cong \mathrm{Ext}^i_{\mathcal{A}_q}(\theta^*(\mathcal{A}_q), q^*(J)) \\ &= \mathrm{Ext}^i_{\mathcal{A}_q}(\mathcal{A}_q, q^*(J)) = 0 \end{aligned}$$

for $i > 0$, as $\mathcal{A}_q$ is projective as an $\mathcal{A}_q$-module. Here we used the fact that θ_* is exact and preserves injectives in the second isomorphism.

(ii) Pick an injective resolution $M \to I^\bullet$. Then, by (i), $p^*(M) \to p^*(I^\bullet)$ is a Γ-acyclic resolution of $p^*(M)$, hence it computes the cohomology of Γ. The result now follows.

(iii) Pick an injective resolution $M \to I^\bullet$ in $\mathscr{C}_R$. Since p_* preserves injectives, it follows that $M \to I^\bullet$ is also an injective resolution in $\mathcal{B}_q$-comod. The result follows.

(iv) Let $M \in \mathscr{C}_R$. By (iii), $R^i\Gamma(M) \cong R^i\tilde{\Gamma}(p_*(M))$ for all $i \geq 0$ and it suffices to show that the right hand side vanishes for $i > N$. To simplify notation we will drop the p_* when referring to an element of $\mathscr{C}_R$ viewed only as a comodule.

Now, note that there is a $\mathcal{B}_q$-comodule map $M \to p^*(M) = \mathcal{A}_q \otimes_R M$ given by $m \mapsto 1 \otimes m$. This map has a splitting given by the $\mathcal{A}_q$-action map, which is a comodule homomorphism by definition of $\mathscr{C}_R$. So, as $\mathcal{B}_q$-comodules, M is a direct summand of

$\mathcal{A}_q \otimes_R M$. This in turn implies that $R^i\tilde{\Gamma}(M)$ is a direct summand of $R^i\tilde{\Gamma}(p^*(M))$. By (ii) the latter equals $R^i\mathrm{Ind}(M)$. Now the result follows from Theorem 2.5.15(iv). □

Definition 4.2.9. We let T_P^R be the subgroup of T_P which maps $(U^{\mathrm{res}})^0$ to $R^\times$, where we view T_P as the group of characters of U_q^0. Note that for $\lambda \in P$, the associated element of T_P belongs in T_P^R. For each $\lambda \in T_P^R$ we have a rank 1 $U^{\mathrm{res}}(\mathfrak{b})$-module R_λ.

When $\lambda \in P$ we may view it as a comodule with coaction $1 \mapsto 1 \otimes \lambda$. In that case, we let $\mathcal{A}_q(\lambda) := p^*(R_{-\lambda})$, which we call a line bundle. More generally, for $M \in \mathscr{C}_R$, we will write $M(\lambda)$ for $M \otimes_R R_{-\lambda}$. By letting $\mathcal{A}_q$ act on the left factor and giving it the tensor $\mathcal{B}_q$-coaction, this is also an element of $\mathscr{C}_R$.

Corollary 4.2.10. *For all $\lambda \in P$ and all $i \geq 0$, $R^i\Gamma(\mathcal{A}_q(\lambda))$ is finitely generated as an R-module. Moreover if $\lambda \in P^+$ then $R^i\Gamma(\mathcal{A}_q(\lambda)) = 0$ for all $i > 0$.*

Proof. By Proposition 4.2.8(ii), we have that $R^i\Gamma(\mathcal{A}_q(\lambda)) = R^i\,\mathrm{Ind}(R_{-\lambda})$. The result now follows from Theorem 2.5.15(i)&(iii). □

We now show results analogous to [13, Lemmas 3.13 & 3.16, Proposition 3.5]. The proofs are essentially identical with the exception of part (ii) of the Lemma below where a few small adjustments are necessary to deal with torsion.

Suppose that M is a $\mathcal{B}_q$-comodule or in other words an integrable $U^{\mathrm{res}}(\mathfrak{b})$-module. We will write V to denote the underlying R-module of M equipped with the trivial $\mathcal{B}_q$-coaction.

Lemma 4.2.11. *Let M be as above.*

(i) If M is in fact an $\mathcal{A}_q$-comodule, viewed as a $\mathcal{B}_q$-comodule via restriction, then $p^*(M) \cong p^*(V)$ in $\mathscr{C}_R$.

(ii) Suppose now that M is finitely generated over R, and moreover suppose that all the weight spaces of M have weight of the form $-\lambda$ where $\lambda \in P^+$. Then

(1) M is acyclic with respect to the induction functor;

(2) there is an R-finite $\mathcal{A}_q$-comodule which surjects onto M as a $\mathcal{B}_q$-comodule.

Proof. (i) We have $p^*(M) = \mathcal{A}_q \otimes_R M$ and $p^*(V) = \mathcal{A}_q \otimes_R V$, which are the same as R-modules. The isomorphism is given by the map $a \otimes m \mapsto \sum am_2 \otimes m_1$ where $m \mapsto \sum m_1 \otimes m_2$ denotes the $\mathcal{A}_q$-coaction. It quite evidently is an $\mathcal{A}_q$-module map, and it is straightforward to check that it is also a $\mathcal{B}_q$-comodule map. Thus this is a morphism in $\mathscr{C}_R$. Quite similarly we have a map going the other way given by $a \otimes m \mapsto \sum aS(m_2) \otimes m_1$, which is also a morphism in $\mathscr{C}_R$ by the Hopf algebra axioms. It also follows from the Hopf algebra axioms that these two maps are inverse to each other, and so we have an isomorphism.

(ii) Write $M = \oplus_\lambda M_{-\lambda}$ for the weight space decomposition of M, where $\lambda \in P^+$ ranges through the weights of M. Since M is finitely generated there are only finitely many weights, and we may list them as $-\lambda_1, -\lambda_2, \ldots, -\lambda_r$ so that $-\lambda_r$ is maximal among them. Hence $N := M_{-\lambda_r}$ is a $U^{\mathrm{res}}(\mathfrak{b})$-submodule. We prove (1) by induction on r. Simply note that N is acyclic by Theorem 2.5.15(iii), and by taking the long exact sequence associated to the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

we see that M is also acyclic by induction hypothesis.

For (2), note that $\mathrm{Ind}(M)$ is finitely generated over R by Theorem 2.5.15(i). Hence the result will follow if we show that the map $\mathrm{Res}\,\mathrm{Ind}(M) \to M$ coming from Frobenius reciprocity is surjective. We prove this by induction on r. Suppose that $r = 1$ so that M is isomorphic to a finite direct sum of modules all of the form $R_{-\lambda}$ or $R_{-\lambda}/\pi^n R_{-\lambda}$ for some $n \geq 1$. Then, it suffices to prove the claim for these summands. The map $\mathrm{Ind}(R_{-\lambda}) \to R_{-\lambda}$ is just projection onto the $(-\lambda)$-weight space, hence the claim follows for $R_{-\lambda}$ by Theorem 2.5.15(ii). It then follows that it also true for any $R_{-\lambda}/\pi^n R_{-\lambda}$ since we have a commutative diagram

$$\begin{tikzcd}
R_{-\lambda} \arrow[r, twoheadrightarrow] & R_{-\lambda}/\pi^n R_{-\lambda} \\
\mathrm{Res}\,\mathrm{Ind}(R_{-\lambda}) \arrow[u, twoheadrightarrow] \arrow[r] & \mathrm{Res}\,\mathrm{Ind}(R_{-\lambda}/\pi^n R_{-\lambda}) \arrow[u]
\end{tikzcd}$$

Now for $r > 1$ we consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & M \arrow[r] & M/N \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Res}\,\mathrm{Ind}(N) \arrow[u] \arrow[r] & \mathrm{Res}\,\mathrm{Ind}(M) \arrow[u] \arrow[r] & \mathrm{Res}\,\mathrm{Ind}(M/N) \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

in which both rows are exact by (1). We then conclude that $\mathrm{Res}\,\mathrm{Ind}(M) \to M$ is surjective by the induction hypothesis and the Five Lemma. $\square$

Let $\mathrm{coh}(\mathscr{C}_R)$ denote the full subcategory of $\mathscr{C}_R$ consisting of objects M which are finitely generated as $\mathcal{A}_q$-modules. We call elements of $\mathrm{coh}(\mathscr{C}_R)$ *coherent modules.*

Proposition 4.2.12. *Let $M \in \mathrm{coh}(\mathscr{C}_R)$. Then there exists $\lambda \in P^+$ such that for all $\mu \in \lambda + P^+$, $M(\mu)$ is generated by finitely many global sections. In particular there is finite direct sum of $\mathcal{A}_q(-\lambda)$ surjecting onto M in $\mathscr{C}_R$.*

Proof. Suppose $m_1, \ldots, m_n$ generate M over $\mathcal{A}_q$. Since M is a $\mathcal{B}_q$-comodule, i.e. an integrable $U^{\mathrm{res}}(\mathfrak{b})$-module, it is in particular locally finite. So if we let W denote the $U^{\mathrm{res}}(\mathfrak{b})$-submodule they generate, then we have that W is finitely generated over R. Moreover we have a surjection $p^*(W) \to M$ in $\mathscr{C}_R$. We may pick $\lambda \in P$ such that $W(\lambda) = W \otimes_R R_{-\lambda}$ satisfies the conditions of Lemma 4.2.11(ii) and let N be an R-finite $\mathcal{A}_q$-comodule surjecting onto $W(\lambda)$. Then $p^*(N)$ surjects onto $p^*(W(\lambda))$ and hence onto $M(\lambda)$. By Lemma 4.2.11(i) and since N is finite over R, we have that $p^*(N)$ is generated as an $\mathcal{A}_q$-module by finitely many global sections, and these define a surjection $\mathcal{A}_q^r \to p^*(N)$. Thus we have a surjection $\mathcal{A}_q^r \to M(\lambda)$ and twisting by $-\lambda$ we get a surjection $\oplus_{i=1}^r \mathcal{A}_q(-\lambda) \to M$ as claimed. Of course the same argument shows that $M(\mu)$ is generated by finitely many global sections for any $\mu \in \lambda + P^+$. $\square$

We will soon repeatedly use a general construction, which we record here:

Lemma 4.2.13. *Let $M \in \mathscr{C}_R$ and let $m_1, \dots, m_i \in M$ for some $i \geq 1$. Then there is a unique minimal coherent submodule P of M such that $m_1, \dots, m_i \in P$.*

Proof. Let N be the $U^{\mathrm{res}}(\mathfrak{b})$-submodule of M generated by $m_1, \dots, m_i$. Then N is R-finite and we let P be the $\mathcal{A}_q$-submodule of M generated by N. Since the $\mathcal{A}_q$-action on M is a comodule homomorphism it follows that P is a subcomodule of M and it is in $\mathrm{coh}(\mathscr{C}_R)$ as N is finite over R. Moreover, any coherent submodule of M which contains $m_1, \dots, m_i$ must also contain N, and so must contain P. $\square$

Since we do not know whether $\mathcal{A}_q$ is Noetherian or not, it is not clear yet that $\mathrm{coh}(\mathscr{C}_R)$ is a well-behaved category. This is what we turn to next. We first need to establish:

Lemma 4.2.14. *The ring R_q is graded Noetherian.*

Proof. Since $q^{\frac{1}{d}} \equiv 1 \pmod{\pi}$, the U_k^{res}-representation $\Gamma(\mathcal{A}_q(\lambda)) \otimes_R k$ is just the global sections of the usual line bundle $\mathcal{L}_\lambda$ on the flag variety G_k/B_k over k for any $\lambda \in P^+$ by Proposition 2.5.18(iii), where we note that $\mathcal{L}_\lambda$ has no higher cohomology by the classical Kempf vanishing theorem (see e.g. [46, Proposition II.4.5]). Hence we see that the ring $R_q/\pi R_q$ is isomorphic to the ring of regular functions on the basic affine space G_k/N_k, and so is Noetherian. Moreover the graded pieces $\Gamma(\mathcal{A}_q(\lambda))$ are all finitely generated over R by Corollary 4.2.10. Thus the result follows from Proposition 2.6.2(ii). $\square$

Theorem 4.2.15. *The modules in $\mathscr{C}_R$ which are finitely generated as $\mathcal{A}_q$-modules coincide exactly with the Noetherian objects.*

Proof. We prove this result in several steps. First we claim that $\mathcal{A}_q$ satisfies ACC in the category $\mathrm{coh}(\mathscr{C}_R)$. Indeed, assume we have a chain

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

of coherent submodules of $\mathcal{A}_q$. Recall the functor $\underline{\Gamma}$ from Remark 4.2.6. By Noetherianity of $R_q = \underline{\Gamma}(\mathcal{A}_q)$ and by left exactness of $\underline{\Gamma}$, we get that there is some $m \geq 1$ such that for all $n \geq m$, $\underline{\Gamma}(M_n) = \underline{\Gamma}(M_m)$. In particular we get that $\Gamma(M_n(\lambda)) = \Gamma(M_m(\lambda))$ for all $\lambda \in P^+$. Fix any $n \geq m$. Then by Proposition 4.2.12, we may pick $\lambda >> 0$ such that both $M_n(\lambda)$ and $M_m(\lambda)$ are generated by their global sections. But then the above equality of global sections implies that $M_n(\lambda) = M_m(\lambda)$ and hence after untwisting that $M_n = M_m$.

Next, we claim that $\mathcal{A}_q$ satisfies ACC in $\mathscr{C}_R$. Indeed, suppose we have a chain

$$M_1 \subset M_2 \subset M_3 \subset \cdots$$

of subobjects of $\mathcal{A}_q$ with $M_i \neq M_{i+1}$ for every $i \geq 1$. Then we may pick $m_1 \in M_1$ and $m_i \in M_i \setminus M_{i-1}$ for every $i \geq 2$. By Lemma 4.2.13, for each $i \geq 1$ we may consider the smallest coherent submodule P_i of M_i which contains $m_1, \dots, m_i$. Note that $P_i \subset P_{i+1}$ by the proof of Lemma 4.2.13. But $m_i \in P_i$ for every i, so that we get a strict ascending chain

$$P_1 \subset P_2 \subset P_3 \subset \cdots$$

of coherent submodules of $\mathcal{A}_q$, which is a contradiction by our first step.

Thus we have proved that $\mathcal{A}_q$ is a Noetherian object. It is then immediate that every line bundle $\mathcal{A}_q(\lambda)$ is also a Noetherian object. But by Proposition 4.2.12, this implies that every coherent module is a Noetherian object. Finally, for the converse, the above argument that $\mathcal{A}_q$ satisfies ACC in $\mathscr{C}_R$ also shows that Noetherian objects are finitely generated over $\mathcal{A}_q$. Indeed, if M is not finitely generated, pick $m_1 \in M$ and let P_1 be the smallest coherent submodule of M containing m_1, given by Lemma 4.2.13. Since M is not coherent we have that $M \neq P_1$. So we can pick $m_2 \in M$ such that $m_2 \notin P_1$. Then we may apply Lemma 4.2.13 again and set P_2 to be the smallest coherent submodule of M containing m_1, m_2. By construction, $P_1 \subset P_2$ is a strict inclusion. As M is not coherent, we may pick $m_3 \in M \setminus P_2$. Carrying on, we get a strict ascending chain

$$P_1 \subset P_2 \subset P_3 \subset \cdots$$

so that M is not a Noetherian object. $\square$

This in particular shows that $\mathrm{coh}(\mathscr{C}_R)$ is an abelian category. This has a few consequences:

Proposition 4.2.16. *Let $M \in \mathrm{coh}(\mathscr{C}_R)$. Then:*

(i) there exists $\lambda \in P^+$ such that for all $\mu \in \lambda + P^+$, $M(\mu)$ is acyclic; and

(ii) (Serre finiteness) *for all $i \geq 0$, $R^i\Gamma(M)$ is finitely generated as an R-module.*

Proof. (i) By Proposition 4.2.12 and Theorem 4.2.15, we may find a resolution of M of the form

$$F_\bullet : F_N \xrightarrow{f_N} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} M \to 0$$

where the F_i are finite direct sums of line bundles. Pick $\lambda \in P$ sufficiently large such that all the line bundles in $F_\bullet(\lambda)$ are of the form $\mathcal{A}_q(\mu)$ for $\mu \in P^+$. Then by Corollary 4.2.10, all the $F_i(\lambda)$ are Γ-acyclic. Let $K_0 = M(\lambda)$ and $K_j = \ker f_j(\lambda)$ for $1 \leq j \leq N$. Then we have a short exact sequence

$$0 \to K_j \to F_j(\lambda) \xrightarrow{f_j(\lambda)} K_{j-1} \to 0$$

for every $1 \leq j \leq N$, and the long exact sequence yields isomorphisms $R^i\Gamma(K_{j-1}) \cong R^{i+1}\Gamma(K_j)$ for all $i \geq 1$. Thus, by using Proposition 4.2.8(iv), we obtain

$$R^i\Gamma(M(\lambda)) \cong R^{i+1}\Gamma(K_1) \cong \cdots \cong R^{i+N}\Gamma(K_N) = 0$$

for all $i \geq 1$ as required. Again the same argument works by replacing λ by any $\mu \in \lambda + P^+$.

(ii) The proof we give is completely analogous to the classical proof for projective schemes in [41, Theorem III.5.2]. First note that by Proposition 4.2.8(iv), we have $R^i\Gamma(M) = 0$ for all $i > N$ and so we may assume that $i \leq N$. We will prove the result by downwards induction on i, the cases $i > N$ being already covered.

By Proposition 4.2.12 there is a surjection $f : \bigoplus_{j=1}^n \mathcal{A}_q(-\lambda_j) \to M$ in $\mathscr{C}_R$, where each $\lambda_j \in P^+$. This gives a short exact sequence

$$0 \to K \to \bigoplus_{j=1}^{n} \mathcal{A}_q(-\lambda_j) \to M \to 0$$

Applying the long exact sequence, we obtain

$$\cdots \to \bigoplus_{j=1}^{n} R^i\Gamma(\mathcal{A}_q(-\lambda_j)) \to R^i\Gamma(M) \to R^{i+1}\Gamma(K) \to \cdots$$

By the induction hypothesis applied to K (which we may apply by Theorem 4.2.15), we get that $R^{i+1}\Gamma(K)$ is finitely generated. Now by Corollary 4.2.10 and since R is Noetherian, we see that $R^i\Gamma(M)$ is finitely generated over R as well. □

One of our main aims will be to establish a D-modules version of Proposition 4.2.16(i). Before we get to that, we can now finally fulfill our promise:

Proof of Theorem 4.2.7. Note that every object of $\mathscr{C}_R$ is a direct limit of objects of $\mathrm{coh}(\mathscr{C}_R)$. Indeed, it suffices to show that every element of any $M \in \mathscr{C}_R$ is contained in a coherent submodule. But this is given by Lemma 4.2.13.

So we just have to check all conditions (i)-(iv) from Proposition 4.2.4. Condition (i) is just Theorem 4.2.15, (ii) follows from the fact that $\Gamma(\mathcal{A}_q) = R$ and from Proposition 4.2.16(ii), and (iii) follows from Proposition 4.2.12. Finally, condition (iv) is easily deduced from Theorem 4.2.15 and Proposition 4.2.16(i). Indeed, suppose $M \to N$ is a surjection between coherent modules in $\mathscr{C}_R$ and let K denote its kernel. For $\lambda >> 0$, we know that $K(\lambda)$ is Γ-acyclic, and so the corresponding map $\Gamma(M(\lambda)) \to \Gamma(N(\lambda))$ is surjective. □

4.3 Weyl group translates of the big cell and Čech complexes

We now introduce certain localisations of $\mathcal{A}_q$ from Joseph (see [48, 3.1-3.3] and [49, 9.1.10]). For each fundamental weight ϖ_i, consider the highest weight representation $V(\varpi_i)$ of U_q. It contains a free R-lattice $M := \mathrm{Ind}(R_{\varpi_i})^*$ that is a U^{res}-module. In fact M is a cyclic module generated by a highest weight vector $v \in V(\varpi_i)$ (see [4, Proposition 3.3]). Let $f \in M^*$ be the corresponding dual vector. Let $c_{\varpi_i} := c^M_{f,v} \in \mathcal{A}_q$ be the corresponding matrix coefficient. Joseph showed in *loc. cit.* that these commute and we may define for any $\mu = \sum_i n_i\varpi_i \in P^+$ the element $c_\mu = \prod_i c_{\varpi_i}^{n_i} \in \mathcal{A}_q$. Moreover, for any $\mu \in P^+$, $c_\mu = c^{V(\mu)}_{f_\mu,v_\mu}$ is a matrix coefficient of the highest weight representation $V(\mu)$ of U_q. In fact it is a matrix coefficient of a U^{res}-lattice inside $V(\mu)$, namely $\mathrm{Ind}(R_{-\mu})^*$.

Recall that $\mathcal{A}_q$ is a U^{res}-module algebra via the action $u \cdot f = \sum f_2(u) f_1$. If we identify $\mathcal{A}_q$ with a submodule of $\mathrm{Hom}_R(U^{\mathrm{res}}, R)$, this action is given by

$$(u \cdot f)(x) = f(xu)$$

for all $u, x \in U^{\mathrm{res}}$ and all $f \in \mathcal{A}_q$. Therefore, identifying c_μ with the matrix coefficient corresponding to a highest weight vector as above, we see that $u \cdot c_\mu = \mu(u) c_\mu$ for any $u \in (U^{\mathrm{res}})^0$ and $E_{\alpha_i}^{(r)} \cdot c_\mu = 0$ for any i and any $r \geq 1$. Thus in the $\mathcal{B}_q$-comodule language, we have $\Delta(c_\mu) = c_\mu \otimes \mu \in \mathcal{A}_q \otimes \mathcal{B}_q$. So we see that $c_\mu \in \Gamma(\mathcal{A}_q(\mu))$.

Recall now that $\Gamma(\mathcal{A}_q(\mu)) = \mathrm{Ind}\, R_{-\mu}$ is an integrable U^{res}-module. The elements of it can all be identified as certain functions in $\mathrm{Hom}_R(U^{\mathrm{res}}, R)$, and the module structure is given by

$$(u \cdot f)(x) = f(S(u)x)$$

for all $u, x \in U^{\mathrm{res}}$ and all $f \in \Gamma(\mathcal{A}_q(\mu))$.

Remark 4.3.1. Note that this is seemingly different from the action given in Definition 2.5.14. This is because we've already identified $\mathrm{Ind}\, R_{-\mu}$ with $(\mathcal{A}_q \otimes_R R_\lambda)^{U^{\mathrm{res}}(\mathfrak{b})}$ where the invariants are taken with respect to the tensor action (as stated in ibid). After having done this identification, and identifying $\mathcal{A}_q \otimes_R R_\lambda$ with an R-submodule of $\mathrm{Hom}_R(U^{\mathrm{res}}, R)$, the U^{res}-action on $\mathrm{Ind}\, R_{-\mu}$ becomes the one given above. This may be viewed in analogy to the induction functor for algebraic groups, which can be defined either as

$$\mathrm{Ind}_B^G(M) = \{f : G \to M : f(gb) = b^{-1} f(g)\}$$

or as

$$\mathrm{Ind}_B^G(M) = \{f : G \to M : f(b^{-1}g) = b^{-1} f(g)\},$$

where M is a B-module and $g \in G$, $b \in B$. Then the G-module structures are given respectively by $(g \cdot f)(g') = f(g^{-1}g')$ and $(g \cdot f)(g') = f(g'g)$.

With respect to this action on $\Gamma(\mathcal{A}_q(\mu))$, the element c_μ has weight $-\mu$ and so is a lowest weight vector, since the module $\Gamma(\mathcal{A}_q(\mu))$ is a free R-lattice inside $V(-w_0\mu)$ and satisfies the Weyl character formula by Theorem 2.5.15(ii). In particular we see that $\Gamma(\mathcal{A}_q(\mu))$ has a unique (up to scalars) extreme w-weight vector $c_{w\mu}$ of weight $-w\mu$ for any Weyl group element $w \in W$, which we may choose to equal

$$c_{w\mu} = E_{\alpha_{i_1}}^{(r_1)} \cdots E_{\alpha_{i_s}}^{(r_s)} \cdot c_\mu$$

where $w = s_{i_1} \cdots s_{i_s}$ and where the exponents r_j are defined by $r_s = \langle \mu, \alpha_{i_s}^\vee \rangle$ and $r_j = \langle s_{i_{j+1}} \cdots s_{i_s} \mu, \alpha_{i_j}^\vee \rangle$ for $j \leq s-1$. Then Joseph [49, 9.1.10] showed that $c_{w\lambda} c_{w\mu} = c_{w(\lambda+\mu)}$ for every $w \in W$ and every $\lambda, \mu \in P^+$. Therefore, for every $w \in W$, the set

$$S_w := \{c_{w\mu} : \mu \in P^+\}$$

is multiplicatively closed in $\mathcal{A}_q$. Moreover we still have $c_{w\mu} \in \Gamma(\mathcal{A}_q(\mu))$, so that we may view S_w as a multiplicatively closed subset of R_q. Joseph showed in *loc. cit.* that S_w is an Ore set in both $\mathcal{O}_q$ and its representation ring, but in fact his proof works equally well with integral forms and so with $\mathcal{A}_q$ and in R_q (this was already pointed out in [56, III.2]). Hence we have:

Lemma 4.3.2. *For every $w \in W$, S_w is an Ore set in $\mathcal{A}_q$ and in R_q.*

So we may define localisations $\mathcal{A}_{q,w} := S_w^{-1}\mathcal{A}_q$ for each Weyl group element. By viewing $\mathcal{A}_q \otimes \mathcal{B}_q$ as a left $\mathcal{A}_q$-module via the comultiplication, the comodule map $\Delta : \mathcal{A}_q \to \mathcal{A}_q \otimes \mathcal{B}_q$ is by definition an $\mathcal{A}_q$-module map, and its localisation gives a map

$$\Delta_w : \mathcal{A}_{q,w} \to \mathcal{A}_{q,w} \otimes_R \mathcal{B}_q$$

which defines a $\mathcal{B}_q$-comodule structure: for $f \in \mathcal{A}_q$ and $s \in S_w$ such that $\Delta(s) = s \otimes \lambda$, Δ_w sends $s^{-1}f$ to $(s^{-1} \otimes -\lambda) \cdot \Delta(f)$. Moreover the $\mathcal{A}_{q,w}$-module structure on $\mathcal{A}_{q,w} \otimes_R \mathcal{B}_q$ is defined by Δ_w.

More generally, if $M \in \mathscr{C}_R$ with comodule map $\rho : M \to M \otimes_R \mathcal{B}_q$ then, by the axioms for $\mathscr{C}_R$, ρ is an $\mathcal{A}_q$-module map where we view $M \otimes_R \mathcal{B}_q$ as an $\mathcal{A}_q$-module via Δ, and its localisation gives rise to a map

$$\rho_w : S_w^{-1}M \to S_w^{-1}M \otimes_R \mathcal{B}_q$$

which will be $\mathcal{A}_{q,w}$-linear where $\mathcal{A}_{q,w}$ acts on $S_w^{-1}M \otimes_R \mathcal{B}_q$ via the map Δ_w.

Definition 4.3.3. We define $\mathscr{C}_R^w$ to be the category of $\mathcal{B}_q$-equivariant $\mathcal{A}_{q,w}$-modules. Specifically, the objects consist of $\mathcal{A}_{q,w}$-modules M which are equipped with a right $\mathcal{B}_q$-comodule structure $M \to M \otimes_R \mathcal{B}_q$ such that the $\mathcal{A}_{q,w}$-action map $\mathcal{A}_{q,w} \otimes_R M \to M$ is a comodule homomorphism where we give $\mathcal{A}_{q,w} \otimes_R M$ the tensor comodule structure. The morphisms are just the $\mathcal{A}_{q,w}$-linear maps which are also comodule homomorphisms.

The above discussion shows that there is a localisation functor $f_w^* : \mathscr{C}_R \to \mathscr{C}_R^w$ which sends a module M to its localisation $S_w^{-1}M$ as an $\mathcal{A}_q$-module, and it has a right adjoint $(f_w)_*$ given by the forgetful functor. We note that both of these are exact and they make $\mathscr{C}_R^w$ into a localisation of $\mathscr{C}_R$ in the sense of Gabriel i.e. a quotient of $\mathscr{C}_R$ by a localising subcategory (see [38, Chapter III.2]).

Remark 4.3.4. If we set $\mathcal{O}_{q,w} := \mathcal{A}_{q,w} \otimes_R L$ for each Weyl group element $w \in W$, then all our constructions can analogously be made for $\mathcal{O}_{q,w}$. In particular, we may completely analogously define the category $\mathcal{M}_{B_q}(G_q)_w$ of $\mathcal{O}_{q,w}$-modules which are also $\mathcal{O}_q(B)$-comodules such that the $\mathcal{O}_{q,w}$-action is a comodule homomorphism. The ring $\mathcal{O}_{q,w}$ is a localisation of $\mathcal{O}_q$ and the category $\mathcal{M}_{B_q}(G_q)_w$ is a localisation of $\mathcal{M}_{B_q}(G_q)$.

Next, we describe the above constructions in the language of proj categories. We saw that $\mathscr{C}_R$ is equivalent to $\mathrm{Proj}(R_q)$ and that we may equally localise any graded R_q-module at the set S_w for any $w \in W$. Since the set S_w contains elements of arbitrarily large degree in R_q, we see that the localisation functor R_q-mod $\to S_w^{-1}R_q$-mod factors through $\mathrm{Proj}(R_q)$ and makes $S_w^{-1}R_q$-mod into a localisation of $\mathrm{Proj}(R_q)$.

We have a global section functor on $\mathscr{C}_R^w$ which corresponds to taking $\mathcal{B}_q$-invariants. This is of course the same as the composite $\Gamma \circ (f_w)_*$. Now via the proj construction we see that global sections on $\mathscr{C}_R$ correspond to projection onto the degree 0 in $\mathrm{Proj}(R_q)$. So we see that the global section functor on $S_w^{-1}R_q$-mod is the functor of taking the degree 0 part of the graded module, which is exact! We then get:

Lemma 4.3.5. *The categories $S_w^{-1}R_q$-mod and $\mathscr{C}_R^w$ have enough injectives, and they are naturally equivalent to each other as localisations of $\mathscr{C}_R$. Hence the global section functor on $\mathscr{C}_R^w$ is exact and objects of $\mathscr{C}_R^w$ are acyclic when viewed in $\mathscr{C}_R$.*

Proof. Since the category $\mathscr{C}_R$ has enough injectives by Lemma 4.2.2, it is then well-known that any localisation of it also has enough injectives (see [38, Corollary III.3.2]). So the first part follows. By the above discussion, if the two categories are equivalent then global sections is exact. To prove that $S_w^{-1}R_q$-mod and $\mathscr{C}_R^w$ are equivalent, we just need to show that $M \in \mathscr{C}_R$ has localisation zero if and only if $\underline{\Gamma}(M)$ has localisation zero.

Clearly if $M \in \mathscr{C}_R$ has localisation zero, then so does $\underline{\Gamma}(M)$. Conversely if $\underline{\Gamma}(M)$ has localisation zero, we show that $\underline{\Gamma}(S_w^{-1}M) = 0$, which implies that $S_w^{-1}M = 0$. Indeed suppose $s \in S_w$, $m \in M$ such that $\Delta(s) = s \otimes \mu$ and $s^{-1}m \in \Gamma(S_w^{-1}M(\lambda))$ for some λ. Then

$$\rho(m) = \rho(s(s^{-1}m)) = (s \otimes \mu)\rho(s^{-1}m) = (s \otimes \mu)(s^{-1}m \otimes \lambda) = m \otimes (\lambda + \mu)$$

so that $m \in \Gamma(M(\lambda+\mu))$. By assumption there exists $t \in S_w$ such that $tm = 0$. But then that means that the image of m in $S_w^{-1}M$ is zero and so $s^{-1}m = 0$. Thus we see that $\underline{\Gamma}(S_w^{-1}M) = 0$ as required.

Finally, let $M \in \mathscr{C}_R^w$ and $M \to I^\bullet$ be an injective resolution of M in $\mathscr{C}_R^w$. Note that since $(f_w)_*$ preserves injectives as it is the right adjoint to an exact functor, we have that $(f_w)_*(I^\bullet)$ is an injective resolution of $(f_w)_*(M)$ in $\mathscr{C}_R$, and applying global sections and taking cohomology we obtain $R^i\Gamma((f_w)_*(M)) = 0$ for all $i > 0$ since $\Gamma \circ (f_w)_*$ is exact. $\square$

Remark 4.3.6. We think of $\mathscr{C}_R^w$ as being an analogue of the w-translate of the big cell on the flag variety. Indeed the localisation of R_q at the elements $c_{w\lambda}$ for fixed w corresponds classically to the ring of regular functions on the pre-inverse image in the basic affine space G/N of the w-translate of the big cell.

We then think of the above lemma as telling us that each $\mathscr{C}_R^w$ is in some sense affine. Now to such a situation Rosenberg [68, Sections 1 & 2] (see also [56, section III.3]) explained how to write down an analogue of the Čech complex which allows us to compute the cohomology of the functor Γ. Write $W = \{w_1, \dots, w_m\}$, let $J = \{1, \dots, m\}$ and for each $i \in J$ let $\sigma_i := (f_{w_i})_* \circ f_{w_i}^*$. Moreover for any $\mathbf{i} = (i_1, \dots, i_n) \in J^n$, let $\sigma_{\mathbf{i}} = \sigma_{i_1} \circ \cdots \circ \sigma_{i_n}$. Then by [68, 1.2 & 1.3] we may write down a complex

$$C^{\mathrm{aug}} : \mathrm{id}_{\mathscr{C}_R} \to \bigoplus_{i \in J} \sigma_i \to \bigoplus_{\mathbf{i} \in J^2} \sigma_{\mathbf{i}} \to \bigoplus_{\mathbf{i} \in J^3} \sigma_{\mathbf{i}} \to \cdots \tag{4.3}$$

where the maps are given as follows. Denote the adjunction morphism $\mathrm{id}_{\mathscr{C}_R} \to \sigma_i$ by η_i. Then for any $\mathbf{i} \in J^n$ and any $1 \le j \le n$, there is a natural transformation

$$\xi_n^j : \sigma_{i_1} \circ \cdots \circ \sigma_{i_n} \to \oplus_{i \in J} \sigma_{i_1} \circ \cdots \circ \sigma_{i_{j-1}} \circ \sigma_i \circ \sigma_{i_j} \circ \cdots \circ \sigma_{i_n}$$

given by $\xi_n^j = \oplus_{i \in J} \sigma_{i_1} \cdots \sigma_{i_{j-1}} \eta_i \sigma_{i_j} \cdots \sigma_{i_n}$. The differential in the complex is then given by taking the alternating sum (over all j) of these ξ_n^j.

We may post-compose C^{aug} with the functor of taking global sections to obtain a complex $\check{C}^{\text{aug}}$ called the *augmented standard complex* of Γ. We may also consider the complex

$$C : \bigoplus_{i \in J} \sigma_i \to \bigoplus_{\mathbf{i} \in J^2} \sigma_{\mathbf{i}} \to \bigoplus_{\mathbf{i} \in J^3} \sigma_{\mathbf{i}} \to \cdots$$

and $\check{C} = \Gamma \circ C$, which we call the *standard complex*. We then have:

Proposition 4.3.7. *For any $M \in \mathscr{C}_R$, the complex $C^{aug}(M)$ is exact. Moreover, for $i \geq 0$, the i-th cohomology of the complex $\check{C}(M)$ is isomorphic to $R^i\Gamma(M)$.*

Proof. In this general setup, it was shown by Rosenberg (see [68, Proposition 1.4 & Theorem 2.2]) that the complex computes the cohomology of any left exact functor if the categories $\mathscr{C}_R^{w_i}$ cover the category $\mathscr{C}_R$, meaning that a morphism g in $\mathscr{C}_R$ is an isomorphism if and only if $f_{w_i}^*(g)$ is an isomorphism for all $i \in J$, and if each $\mathscr{C}_R^{w_i}$ has enough injectives (this is Lemma 4.3.5).

So we just need to show that we have a covering. This is equivalent to saying that $M \in \mathscr{C}_R$ is zero if and only if all its localisations are zero. Working with proj categories instead, suppose M is a graded R_q-module such that $S_w^{-1}M = 0$ for all w. Pick $m \in M$. Then for all $i \in J$, there exists $\mu_i \in P^+$ such that $c_{w_i \mu_i} m = 0$. Let $\mu = \sum_i \mu_i$. Then for all $w \in W$, $c_{w\mu} m = 0$. But then it follows from the Lemma below that $\Gamma(\mathcal{A}_q(\lambda + \mu))m = 0$ for all $\lambda >> 0$. Since m was arbitrary this implies that M is torsion and so zero in $\text{Proj}(R_q)$. $\square$

Lemma 4.3.8. *Let $\mu \in P^+$. Then for $\lambda >> 0$ we have*

$$\sum_{w \in W} \Gamma(\mathcal{A}_q(\lambda)) c_{w\mu} = \Gamma(\mathcal{A}_q(\lambda + \mu)).$$

Proof. This is proved in [56, Lemma III.3.3] but we reproduce it here. Clearly the left hand side is included in the right hand side, and both sides are finitely generated as R-modules, so by Nakayama it's enough to show that the equality holds modulo π, i.e. that

$$\sum_{w \in W} H^0(\lambda) \overline{c_{w\mu}} = H^0(\lambda + \mu)$$

where $H^0(\lambda)$ denotes the global sections of the line bundle $\mathcal{L}_\lambda$ on the flag variety G_k/B_k. We are then in a classical situation, and the equality will follow from the classical fact that the Weyl group translates of the big cell cover the flag variety of G_k. The equality was proved over $\mathbb{C}$ in [51, Lemma 11]. The argument is the same here in positive characteristic, but for completeness we sketch it.

Firstly, since both sides are finite dimensional over k, to show equality is to show that the dimensions are equal, and so it will suffice to prove that the equality holds after passing to the algebraic closure of k. So without loss of generality, we may assume that $k = \bar{k}$.

Moreover, for any λ' and μ', the natural map $H^0(\lambda') \otimes H^0(\mu') \to H^0(\lambda' + \mu')$ is surjective (see [46, Proposition 14.20]). Thus we may assume that $\lambda = n\mu$ for $n >> 0$.

Now, consider the Weyl module $V = V(-w_0\mu) = H^0(\mu)^*$ and let $v \in V$ have weight $-w_0\mu$. Then the flag variety G_k/B_k maps onto the G_k-orbit of the line kv in the projective space $\mathbb{P}(V)$. If we take the homogeneous cone above this, its algebra of regular functions is a quotient of $S(V^*)$, and in fact is the commutative graded ring $A = \oplus_{n\geq 0} H^0(n\mu)$ (see [46, Proposition 14.22]). The fact that the Weyl group translates of the big cell cover the flag variety now implies that the radical of the ideal of A generated by the elements $\overline{c_{w\mu}}$ is in fact the irrelevant ideal $A_{>0}$. This says that the ideal they generate contains all $H^0(m\mu)$ for m large enough, as required. $\square$

Remark 4.3.9. Since the setup of Rosenberg [68] is very general, we can make a completely identical construction of the (augmented and non-augmented) standard complex over L, working with the categories $\mathcal{M}_{B_q}(G_q)_w$. The version of the above Lemma 4.3.8 over fields was first proved directly by Joseph ([48, Lemma 2.7]), again using a specialisation at $q = 1$ argument and appealing to the paper [51]. Then, the corresponding Čech-like complex obtained that way does compute the cohomology of global sections on $\mathcal{M}_{B_q}(G_q)$ (see [13, Proposition 4.5]).

As an immediate application of the Čech complex, we show how the cohomology of Γ behaves under base change to the field L.

Proposition 4.3.10. *For any $M \in \mathscr{C}_R$ and any $i \geq 0$, we have $R^i\Gamma(M_L) = R^i\Gamma(M) \otimes_R L$.*

Proof. Let $M \in \mathscr{C}_R$ and first assume that $i = 0$. By the universal property of tensor products, we have a commutative diagram

$$\begin{tikzcd} M \arrow[r, "g"] & M \otimes_R L \\ M^{\mathcal{B}_q} \arrow[u] \arrow[r] & M^{\mathcal{B}_q} \otimes_R L \arrow[u, "f"'] \end{tikzcd}$$

of R-modules with injective vertical arrows, and we have to show that $\mathrm{Im}(f) = (M_L)^{\mathcal{B}_q}$. It will be enough show that $(M_L)^{\mathcal{B}_q} \subseteq \mathrm{Im}(f)$, the other inclusion being clear.

Pick $m \in (M_L)^{\mathcal{B}_q}$. Then there is some $a \geq 0$ such that $\pi^a m \in \mathrm{Im}(g)$, i.e. $\pi^a m = m' \otimes 1$ for some $m' \in M$. Now, given that $\rho_{M_L}(m) = m \otimes 1$, and since $\rho_{M_L} = \rho_M \otimes_R L$, we see that $\rho_M(m') - m' \otimes 1 \in M \otimes_R \mathcal{B}_q$ is π-torsion. Hence, there is some $b \geq 0$ such that $\rho_M(\pi^b m') = \pi^b m' \otimes 1$, and thus we get that $\pi^{a+b} m \in \mathrm{Im}(f)$. The result now follows since $\mathrm{Im}(f)$ is an L-vector space.

For $i > 0$, using Proposition 4.3.7, the case $i = 0$ and the fact that $- \otimes_R L$ is exact, we see that $R^i\Gamma(M) \otimes_R L$ is the i-th Čech cohomology group of $M \otimes_R L$. But by Remark 4.3.9 above, this group is equal to $R^i\Gamma(M \otimes_R L)$. $\square$

4.4 D-modules in $\mathscr{C}_R$

We now define an R-form of the ring of quantum differential operators. For $u \in U$, $a \in \mathcal{A}_q$, $i \geq 0$, we have $u(a) = \sum a_2(u) \cdot a_1 \in \mathcal{A}_q$ since $U \subset U^{\mathrm{res}}$. From this, we can immediately see that $\mathcal{A}_q$ is a left U-module algebra. Hence we may form the smash product $\mathcal{D} = \mathcal{A}_q \# U$. Note that $\mathcal{D}$ is π-torsion free as it is equal to $\mathcal{A}_q \otimes_R U$ as an R-module, thus it follows that it is a lattice in $\mathcal{D}_q$.

Proposition 4.4.1. *The algebra $\mathcal{D}/\pi\mathcal{D}$ is Noetherian. Hence so is $\widehat{\mathcal{D}}$.*

Proof. By the above remarks we see that $\mathcal{D}_k := \mathcal{D}/\pi\mathcal{D}$ is the smash product algebra of $\mathcal{A}_q/\pi\mathcal{A}_q \cong \mathcal{O}(G_k)$ (by Proposition 2.5.18(iv)) and $U_k := U/\pi U$. From Proposition 2.5.18(ii), we have that

$$U_k \cong U(\mathfrak{g}_k)[K_\mu : \mu \in P]/(K_{\alpha_i}^2 - 1 : 1 \leq i \leq n).$$

Hence we see that U_k is a finite $U(\mathfrak{g}_k)$-module: it is generated by $K_{\varpi_1}, \dots, K_{\varpi_n}$ as a $U(\mathfrak{g}_k)$-algebra but these satisfy $K_{\varpi_i}^{2m} = 1$ where m is the index of Q in P, i.e. the determinant of the Cartan matrix. Thus $\mathcal{D}_k$ is a finite module over the smash product $\mathcal{O}(G_k)\#U(\mathfrak{g}_k)$. The latter is isomorphic to the ring $\mathcal{D}(G_k)$ of crystalline differential operators on the affine variety G_k by Example 2.1.18 and hence is Noetherian. Thus $\mathcal{D}_k$ is Noetherian as required. The last part follows from Proposition 2.6.2(i). ☐

We now turn to an R-version of the category $\mathcal{D}^\lambda_{B_q}(G_q)$. We first introduce the following notation: we let $U^{\geq 0} = U \cap U_q^{\geq 0}$. It is the R-subalgebra of U generated by all E_{α_i}, all K_μ ($\mu \in P$) and all $[K_{\alpha_i}; 0]_{q_i}$. Note that $U^{\geq 0}$ is a subalgebra of $U^{\text{res}}(\mathfrak{b})$. Moreover, note that the action (4.2) given just before Definition 4.1.7 restricts to an action of U^{res} on $\mathcal{D}$ making it into a U^{res}-module algebra. This is because the adjoint action of U^{res} preserves U.

Definition 4.4.2. Let $\lambda \in T_P^R$. We let $\mathscr{D}^\lambda$ be the category whose objects are triples (M, α, β) where M is an R-module, $\alpha : \mathcal{D} \otimes_R M \to M$ is a left $\mathcal{D}$-module action and $\beta : M \to M \otimes_R \mathcal{B}_q$ is a right $\mathcal{B}_q$-comodule action. The map β induces a left $U^{\text{res}}(\mathfrak{b})$-action on M which we also denote by β. These actions must satisfy:

(i) The $U^{\geq 0}$-actions on $M \otimes_R R_\lambda$ given by $\beta \otimes \lambda$ and $\alpha|_{U^{\geq 0}} \otimes 1$ are equal.

(ii) The map α is $U^{\text{res}}(\mathfrak{b})$-linear with respect to the β-action on M and the action (4.2) on $\mathcal{D}$.

We will write $\text{coh}(\mathscr{D}^\lambda)$ to denote the full subcategory of $\mathscr{D}^\lambda$ consisting of finitely generated $\mathcal{D}$-modules.

There is of course a forgetful functor $\mathscr{D}^\lambda \to \mathscr{C}_R$, and given an object $M \in \mathscr{D}^\lambda$ we let its global sections equal $\Gamma(M)$ where we view M as an object of $\mathscr{C}_R$. By abuse of notation we also denote this global section functor by Γ. Also the functor $M \mapsto M_L$ described earlier restricts to a functor $\mathscr{D}^\lambda \to \mathcal{D}^\lambda_{B_q}(G_q)$.

Note again that condition (i) above can be rephrased into saying that for $M \in \mathscr{D}^\lambda$ and $m \in M$, we have $E_\alpha m = \beta(E_\alpha)m$, $K_\mu m = \lambda(K_\mu)\beta(K_\mu)m$, and

$$[K_\alpha; 0]m = (\lambda([K_\alpha; 0])\beta(K_\alpha) + \lambda(K_\alpha^{-1})\beta([K_\alpha; 0]))m$$

for all simple roots α and $\mu \in P$. In particular if m is a global section then by $\mathcal{B}_q$-coinvariance we must have $E_\alpha m = 0$, $[K_\alpha; 0]m = \lambda([K_\alpha; 0])m$ and $K_\mu m = \lambda(K_\mu)m$. In other words global sections consist of the highest weight vectors of weight λ. So we see that the $\mathcal{D}$-module homomorphisms $\mathcal{D} \to M$ corresponding to global sections factor through the quotient $\mathcal{D}^\lambda = \mathcal{D}/I$ where I is the left ideal generated by

$$\{E_{\alpha_i}, K_\mu - \lambda(K_\mu), [K_{\alpha_i}; 0] - \lambda([K_{\alpha_i}; 0]) : 1 \leq i \leq n, \mu \in P\}.$$

Our aim now is to show that $\mathcal{D}^\lambda \in \mathscr{D}^\lambda$.

We may define a Verma module $\mathcal{M}_\lambda$ for U, namely it is the cyclic U-module with generator v_λ and relations $E_{\alpha_i} v_\lambda = 0$, $K_\mu v_\lambda = \lambda(\mu)v_\lambda$ and $[K_{\alpha_i}; 0]v_\lambda = \lambda([K_{\alpha_i}; 0])v_\lambda$. By the triangular decomposition for U (Remark 3.3.8) and the PBW basis for U^- (Corollary 3.3.7), we see that $\mathcal{M}_\lambda = U^- v_\lambda$ and is a free R-module with basis given by the monomials

$$F_{\beta_1}^{r_1} \cdots F_{\beta_N}^{r_N} v_\lambda$$

and so we also see that it is a lattice in the Verma module M_λ for U_q. Recall that the quotient of the adjoint action of $U_q^{\geq 0}$ gave rise to an integrable module structure on M_λ.

Lemma 4.4.3. *The above adjoint $U^{res}(\mathfrak{b})$-action on M_λ preserves $\mathcal{M}_\lambda$, making it into a $\mathcal{B}_q$-comodule.*

Proof. We first check that it's preserved under the action of $(U^{\mathrm{res}})^0$. But this follows from the fact that the above R-basis for $\mathcal{M}_\lambda$ is a weight basis and that an element of $(U^{\mathrm{res}})^0$ acts on it by one of the characters (2.3) in the discussion following Remark 2.4.13, which is R-valued. Finally we need to check that it is also preserved by the action of all $E_{\alpha_j}^{(r)}$. Note that since $\mathcal{M}_\lambda = U^- v_\lambda$, it is in fact equal to the R-span of all terms of the form $F_{\alpha_{i_1}} \cdots F_{\alpha_{i_s}} v_\lambda$ for $s \geq 0$ and $1 \leq i_1, \ldots, i_s \leq n$, and that these terms are all weight vectors. We show by induction on s that, for any $r \geq 1$,

$$E_{\alpha_j}^{(r)} F_{\alpha_{i_1}} \cdots F_{\alpha_{i_s}} v_\lambda \in \mathcal{M}_\lambda.$$

The case $s = 0$ is obvious since $E_{\alpha_j} v_\lambda = 0$. When $s \geq 1$, we have two cases: 1) $j \neq i_1$, and 2) $j = i_1$. In case 1) we have that E_{α_j} and $F_{\alpha_{i_1}}$ commute so that

$$E_{\alpha_j}^{(r)} F_{\alpha_{i_1}} \cdots F_{\alpha_{i_s}} v_\lambda = F_{\alpha_{i_1}} E_{\alpha_j}^{(r)} F_{\alpha_{i_2}} \cdots F_{\alpha_{i_s}} v_\lambda \in \mathcal{M}_\lambda$$

where we apply the induction hypothesis. So we're left with case 2). By [45, 1.3(6)] we have the following commutator formula:

$$E_{\alpha_j}^{(r)} F_{\alpha_j} = F_{\alpha_j} E_{\alpha_j}^{(r)} + E_{\alpha_j}^{(r-1)} [K_{\alpha_j}; r-1].$$

Thus, since $[K_{\alpha_j}; r-1]$ rescales weight vectors in M_λ by an element of R, it follows by the induction hypothesis that

$$E_{\alpha_j}^{(r)} F_{\alpha_{i_1}} \cdots F_{\alpha_{i_s}} v_\lambda = F_{\alpha_{i_1}} E_{\alpha_j}^{(r)} F_{\alpha_{i_2}} \cdots F_{\alpha_{i_s}} v_\lambda \\ + E_{\alpha_j}^{(r-1)} [K_{\alpha_j}; r-1] F_{\alpha_{i_2}} \cdots F_{\alpha_{i_s}} v_\lambda \in \mathcal{M}_\lambda$$

as required. □

Now since $\mathcal{D}^\lambda = \mathcal{A}_q \otimes_R \mathcal{M}_\lambda$ as an R-module, we identify it with $p^*(\mathcal{M}_\lambda) \in \mathscr{C}_R$. Just as for $\mathcal{D}_q^\lambda$, we then have that $\mathcal{D}^\lambda$ is in fact an object of $\mathscr{D}^\lambda$, and our previous discussion shows that it represents the global section functor on $\mathscr{D}^\lambda$, i.e.

$$\Gamma(M) = \mathrm{Hom}_{\mathscr{D}^\lambda}(\mathcal{D}^\lambda, M)$$

for all $M \in \mathscr{D}^\lambda$.

We now define twists of D-modules. Observe that for $\mu \in P$ and $M \in \mathscr{D}^\lambda$, the left $\mathcal{D}$-action on $M(\mu)$ makes $M(\mu)$ into an element of $\mathscr{D}^{\lambda+\mu}$.

Proposition 4.4.4. *For $\mu = 0$ or $\mu >> 0$, and for any $n \geq 1$, we have that $\mathcal{D}^\lambda(\mu)/\pi^n\mathcal{D}^\lambda(\mu)$ is Γ-acyclic.*

Proof. We first deal with the case $n = 1$. Since we have

$$\mathcal{D}^\lambda(\mu)/\pi\mathcal{D}^\lambda(\mu) = p^*(\mathcal{M}_\lambda(\mu)/\pi\mathcal{M}_\lambda(\mu)),$$

it's enough to show that $\mathcal{M}_\lambda(\mu)/\pi\mathcal{M}_\lambda(\mu)$ is acyclic with respect to the induction functor by Proposition 4.2.8(ii). Now, this can be identified with $U^-/\pi U^- \cong U_k(\mathfrak{n}_k^-)$ as an R-module, and the PBW filtration on $U_k(\mathfrak{n}_k^-)$ (which is not the same as the filtration coming from the height filtration on U^-!) gives rise to a comodule filtration on $\mathcal{M}_\lambda(\mu)/\pi\mathcal{M}_\lambda(\mu)$. The associated graded here is isomorphic to the (twist of the) symmetric algebra $S(\mathfrak{n}_k^*)(\mu)$. It was shown by Andersen and Jantzen (see [2, Theorem 3.6]) that each graded piece of this is acyclic for $\mu = 0$ or $\mu >> 0$ (in fact μ just needs to be strongly dominant).

If $F_i(\mathcal{M}_\lambda(\mu)/\pi\mathcal{M}_\lambda(\mu))$ denotes the i-th filtered piece in this filtration, we then have that $F_0(\mathcal{M}_\lambda(\mu)/\pi\mathcal{M}_\lambda(\mu)) = \mathrm{gr}_0\, \mathcal{M}_\lambda(\mu)/\pi\mathcal{M}_\lambda(\mu)$ is acyclic and, for each $i \geq 1$, there is a short exact sequence

$$0 \to F_{i-1}(\mathcal{M}_\lambda(\mu)/\pi\mathcal{M}_\lambda(\mu)) \to F_i(\mathcal{M}_\lambda(\mu)/\pi\mathcal{M}_\lambda(\mu)) \to \mathrm{gr}_i\, \mathcal{M}_\lambda(\mu)/\pi\mathcal{M}_\lambda(\mu) \to 0$$

which by induction gives us from the long exact sequence that $F_i(\mathcal{M}_\lambda(\mu)/\pi\mathcal{M}_\lambda(\mu))$ is acyclic. Thus, after taking direct limits, we see that $\mathcal{M}_\lambda(\mu)/\pi\mathcal{M}_\lambda(\mu)$ is acyclic as well.

Now for $n \geq 1$, we have a short exact sequence

$$0 \to \mathcal{D}^\lambda(\mu)/\pi\mathcal{D}^\lambda(\mu) \to \mathcal{D}^\lambda(\mu)/\pi^{n+1}\mathcal{D}^\lambda(\mu) \to \mathcal{D}^\lambda(\mu)/\pi^n\mathcal{D}^\lambda(\mu) \to 0$$

where by the above and by induction hypothesis, the two side terms are acyclic. Hence by the long exact sequence the middle term is acyclic. □

As a consequence of this we can obtain a D-modules version of Proposition 4.2.12 which will be useful to us later. We first need a lemma:

Lemma 4.4.5. *Let $M \in \mathrm{coh}(\mathscr{D}^\lambda)$. Then there is an $\mathcal{A}_q$-submodule N of M such that $N \in \mathrm{coh}(\mathscr{C}_R)$ and N generates M as a $\mathcal{D}$-module.*

Proof. Let $m_1, \ldots, m_n$ be a generating set for M as a $\mathcal{D}$-module. Viewing M as an object of $\mathscr{C}_R$, we simply let N be the smallest coherent submodule of M containing $m_1, \ldots, m_n$, as given by Lemma 4.2.13. □

Theorem 4.4.6. *Let $M \in \mathrm{coh}(\mathscr{D}^\lambda)$. Then $M(\mu)$ is generated by finitely many global sections for $\mu >> 0$. Moreover, if $\pi M = 0$ then $M(\mu)$ is also Γ-acyclic for $\mu >> 0$.*

Proof. Let $N \in \mathrm{coh}(\mathscr{C}_R)$ be as in Lemma 4.4.5. Note that $M(\mu) \in \mathrm{coh}(\mathscr{D}_n^{\lambda+\mu})$ for any μ. By Proposition 4.2.12 we see that $N(\mu)$ is generated by finitely many global sections for $\mu >> 0$. Since M is generated by N as a $\mathcal{D}$-module, the first claim follows.

Now assume $\pi M = 0$. Fix any μ_1 such that $M(\mu_1)$ is generated by finitely many global sections. Then we have a surjection $(\mathcal{D}^{\lambda+\mu_1})^a \to M(\mu_1)$ which in fact factors through a surjection $f_1 : (\mathcal{D}^{\lambda+\mu_1}/\pi\mathcal{D}^{\lambda+\mu_1})^a \to M(\mu_1)$. Let $K = \ker f_1$. Note that $K \in \mathrm{coh}(\mathscr{D}^{\lambda+\mu_1})$ by Proposition 4.4.1, and that $\pi K = 0$. So by the above argument applied to K, we can find $\mu_2 >> 0$ and a surjection $f_2 : (\mathcal{D}^{\lambda+\mu_1+\mu_2}/\pi\mathcal{D}^{\lambda+\mu_1+\mu_2})^b \to K(\mu_2)$. Carrying on we obtain $\mu_1, \ldots, \mu_N \in P^+$ and a resolution in $\mathrm{coh}(\mathscr{D}^{\lambda+\mu})$

$$F_N \xrightarrow{f_N} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} M(\mu) \to 0$$

where $\mu = \sum_{j=1}^N \mu_i$ and for $1 \le i \le N$, F_i is a direct sum of finitely many copies of modules of the form $\mathcal{D}^{\lambda+\mu_1+\ldots+\mu_i}(\mu_{i+1} + \ldots + \mu_N) \otimes_R k$. Note that all the F_i are Γ-acyclic by Proposition 4.4.4. Write $K_0 = M(\mu)$ and $K_i = \ker f_i$ for $1 \le i \le N$. Then for each $1 \le i \le N$ we have a short exact sequence

$$0 \to K_i \to F_i \to K_{i-1} \to 0$$

Since F_i is acyclic the long exact sequence implies that $R^j\Gamma(K_{i-1}) \cong R^{j+1}\Gamma(K_i)$ for all $j \ge 1$. Thus we obtain

$$R^j\Gamma(M(\mu)) \cong R^{j+1}\Gamma(K_1) \cong R^{j+2}\Gamma(K_2) \cong \ldots \cong R^{j+N}\Gamma(K_N) = 0$$

for any $j \ge 1$ as required. □

Remark 4.4.7. We expect the above result to hold for all modules, not just for those killed by π.

Chapter 5

Analytic quantum flag varieties and the Beilinson-Bernstein theorem

In this chapter, we define the analytic quantum flag variety and D-modules on it, and we prove Theorems C, D & E from the Introduction. We keep our convention from Chapter 3 that the term "flat" will be used to mean "flat on both sides" unless explicitly stated otherwise.

5.1 Banach $\widehat{\mathcal{O}_q(B)}$-comodules

We first begin by defining a suitable version of comodules over $\widehat{\mathcal{B}_q}$.

Notation. Given two R-modules M and N, we write $M\widehat{\otimes}_R N$ to denote the π-adic completion $\widehat{M \otimes_R N}$ of $M \otimes_R N$. This construction satisfies the usual associativity and additivity properties of tensor products, and is functorial.

Definition 5.1.1. A $\widehat{\mathcal{B}_q}$-comodule is a π-adically complete R-module $\mathcal{M}$ equipped with a map $\rho : \mathcal{M} \to \mathcal{M}\widehat{\otimes}_R\mathcal{B}_q$ such that

$$(\rho\widehat{\otimes}\,\mathrm{id}_{\mathcal{B}_q}) \circ \rho = (\mathrm{id}_{\mathcal{M}}\,\widehat{\otimes}\Delta) \circ \rho, \quad \text{and} \quad (\mathrm{id}_{\mathcal{M}}\,\widehat{\otimes}\varepsilon) \circ \rho = \mathrm{id}_{\mathcal{M}}\,.$$

A morphism of $\widehat{\mathcal{B}_q}$-comodules is an R-module map $f : \mathcal{M} \to \mathcal{N}$ such that $(f\widehat{\otimes}\,\mathrm{id}_{\mathcal{B}_q}) \circ \rho_{\mathcal{M}} = \rho_{\mathcal{N}} \circ f$. We denote the set of comodule morphisms $\mathcal{M} \to \mathcal{N}$ by $\mathrm{Hom}_{\widehat{\mathcal{B}_q}}(\mathcal{M}, \mathcal{N})$.

Lemma 5.1.2. *Suppose that $\mathcal{M}$ is a $\widehat{\mathcal{B}_q}$-comodule. Then $\mathcal{M}/\pi^n\mathcal{M}$ is a $\mathcal{B}_q$-comodule for every $n \geq 1$. Hence $\mathcal{M}$ is a $\widehat{U^{res}(\mathfrak{b})}$-module and, moreover, if ρ_n denotes the $\mathcal{B}_q$-comodule map on $\mathcal{M}/\pi^n\mathcal{M}$ and ρ denotes the $\widehat{\mathcal{B}_q}$-comodule map on $\mathcal{M}$, then $\rho = \varprojlim \rho_n$.*

Proof. There are isomorphisms

$$(\mathcal{M}\widehat{\otimes}_R\mathcal{B}_q)/\pi^n(\mathcal{M}\widehat{\otimes}_R\mathcal{B}_q) \cong (\mathcal{M} \otimes_R \mathcal{B}_q)/\pi^n(\mathcal{M} \otimes_R \mathcal{B}_q) \cong (\mathcal{M}/\pi^n\mathcal{M}) \otimes_R \mathcal{B}_q$$

by Lemma 3.1.3, and hence $\rho : \mathcal{M} \to \mathcal{M}\widehat{\otimes}_R \mathcal{B}_q$ induces a map

$$\rho_n : \mathcal{M}/\pi^n\mathcal{M} \to (\mathcal{M}/\pi^n\mathcal{M}) \otimes_R \mathcal{B}_q$$

for every $n \geq 1$. The comodule axioms are satisfied since they are obtained by reducing the equalities

$$(\rho\widehat{\otimes}\,\mathrm{id}_{\mathcal{B}_q}) \circ \rho = (\mathrm{id}_{\mathcal{M}}\,\widehat{\otimes}\Delta) \circ \rho \quad \text{and} \quad (\mathrm{id}_{\mathcal{M}}\,\widehat{\otimes}\varepsilon) \circ \rho = \mathrm{id}_{\mathcal{M}}$$

modulo π^n. Hence $\mathcal{M}/\pi^n\mathcal{M}$ is a $U^{\mathrm{res}}(\mathfrak{b})$-module and even a $U^{\mathrm{res}}(\mathfrak{b})/\pi^n U^{\mathrm{res}}(\mathfrak{b})$-module, and the structures are compatible with the maps $\mathcal{M}/\pi^{n+1}\mathcal{M} \to \mathcal{M}/\pi^n\mathcal{M}$. Taking inverse limits we see that $\mathcal{M}$ is a $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-module. The last part is immediate since $\varprojlim \rho_n = \widehat{\rho} = \rho$ as $\mathcal{M}$ is π-adically complete. □

Corollary 5.1.3. *For any two $\widehat{\mathcal{B}_q}$-comodules $\mathcal{M}$ and $\mathcal{N}$, there is a canonical isomorphism* $\mathrm{Hom}_{\widehat{\mathcal{B}_q}}(\mathcal{M}, \mathcal{N}) \cong \varprojlim \mathrm{Hom}_{\mathcal{B}_q}(\mathcal{M}/\pi^n\mathcal{M}, \mathcal{N}/\pi^n\mathcal{N})$. *Moreover every $\widehat{\mathcal{B}_q}$-comodule homomorphism is $\widehat{U^{res}(\mathfrak{b})}$-linear.*

Proof. Given a $\widehat{\mathcal{B}_q}$-comodule homomorphism $f : \mathcal{M} \to \mathcal{N}$, the induced map $f_n : \mathcal{M}/\pi^n\mathcal{M} \to \mathcal{N}/\pi^n\mathcal{N}$ is a $\mathcal{B}_q$-comodule map for every $n \geq 1$: since f is a comodule homomorphism we have that $(f\widehat{\otimes}\,\mathrm{id}_{\mathcal{B}_q}) \circ \rho_{\mathcal{M}} = \rho_{\mathcal{N}} \circ f$, which gives that f_n is a comodule homomorphism by reducing modulo π^n. Moreover the maps f_n uniquely determine f since $f = \varprojlim f_n$. Hence this implies that f is $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-linear since the maps f_n are all $U^{\mathrm{res}}(\mathfrak{b})$-linear. All this defines an injective map

$$\mathrm{Hom}_{\widehat{\mathcal{B}_q}}(\mathcal{M}, \mathcal{N}) \to \varprojlim \mathrm{Hom}_{\mathcal{B}_q}(\mathcal{M}/\pi^n\mathcal{M}, \mathcal{N}/\pi^n\mathcal{N})$$

and we need to check that it is surjective. But given an inverse system of maps $f_n : \mathcal{M}/\pi^n\mathcal{M} \to \mathcal{N}/\pi^n\mathcal{N}$, passing to the inverse limit gives rise to a map $f : \mathcal{M} \to \mathcal{N}$ which is a comodule homomorphism since the axioms are satisfied modulo π^n for every $n \geq 1$. □

We now start preparing for an equivalent notion to the notion of $\widehat{\mathcal{B}_q}$-comodules. Note that by Lemma 5.1.2, if $\mathcal{M}$ is a $\widehat{\mathcal{B}_q}$-comodule, then $\mathcal{M}/\pi^n\mathcal{M}$ is an integrable $U^{\mathrm{res}}(\mathfrak{b})$-module for every $n \geq 1$. We want an analogous notion of integrable modules at this π-adically complete level.

Definition 5.1.4. Let $\mathcal{M}$ be a π-adically complete $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-module. Given $\lambda \in P$, we define the λ-weight space $\mathcal{M}_\lambda$ to be the corresponding weight space of $\mathcal{M}$ viewing it as a $U^{\mathrm{res}}(\mathfrak{b})$-module. We say that $\mathcal{M}$ is *topologically integrable* as a $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-module if:

(i) $\mathcal{M}$ is *topologically* $(U^{res})^0$*-semisimple*, i.e. for every $m \in \mathcal{M}$ there exists a family $(m_\lambda)_{\lambda \in P}$ such that $m_\lambda \in \mathcal{M}_\lambda$ and $\sum_{\lambda \in P} m_\lambda$ converges to m in $\mathcal{M}$; and

(ii) for every i the action of E_{α_i} on M is *locally topologically nilpotent*, i.e. for every $m \in \mathcal{M}$ the sequence $E_{\alpha_i}^{(r)} \cdot m \to 0$ as $r \to \infty$.

A morphism of topologically integrable modules is a $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-module map.

Proposition 5.1.5. *Let M be a $U^{res}(\mathfrak{b})$-module and let $\mathcal{M}$ be a $\widehat{U^{res}(\mathfrak{b})}$-module. Then:*

(i) if $\mathcal{M}$ is topologically integrable, then it has a canonical $\widehat{\mathcal{B}_q}$-comodule structure; and

(ii) if M is integrable, then $\widehat{M}$ is a topologically integrable $\widehat{U^{res}(\mathfrak{b})}$-module.

Proof. For (i), note that it follows immediately from the definition of topologically integrable $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-module that $\mathcal{M}/\pi^n\mathcal{M}$ is integrable as a $U^{\mathrm{res}}(\mathfrak{b})$-module for every $n \geq 1$. So there are comodule maps

$$\rho_n : \mathcal{M}/\pi^n\mathcal{M} \to \mathcal{M}/\pi^n\mathcal{M} \otimes_R \mathcal{B}_q \cong (\mathcal{M} \otimes_R \mathcal{B}_q)/\pi^n(\mathcal{M} \otimes_R \mathcal{B}_q)$$

for every $n \geq 1$, which are compatible with the maps $\mathcal{M}/\pi^{a+1}\mathcal{M} \to \mathcal{M}/\pi^a\mathcal{M}$. Taking inverse limits gives a map

$$\rho : \mathcal{M} \to \mathcal{M}\widehat{\otimes}_R\mathcal{B}_q$$

which gives a comodule structure to $\mathcal{M}$: the comodule axioms hold modulo π^n for every $n \geq 1$ so hold for ρ. The module structure arising from ρ agrees by definition with the initial module structure on $\mathcal{M}$.

For (ii), let $m \in \widehat{M}$. Then there exists $m_0, m_1, \ldots$ in M such that

$$m = \sum_{i=0}^{\infty} \pi^i m_i.$$

Now, as M is integrable, we can find an ascending chain of finite subsets $S_j \subseteq P$ such that $m_j = \sum_{\lambda \in S_j} m_{j,\lambda}$ for some $m_{j,\lambda} \in M_\lambda$. Let $S = \bigcup_{j \geq 0} S_j$. For each $\lambda \in S$, let

$$n(\lambda) = \inf\{j : \lambda \in S_j\}.$$

Then set

$$m_\lambda = \sum_{j \geq n(\lambda)} \pi^j m_{j,\lambda} \in \pi^{n(\lambda)}\widehat{M}_\lambda.$$

Since each set S_j is finite, each set $\{\lambda : n(\lambda) < j\}$ is also finite and so $\sum_{\lambda \in S} m_\lambda$ converges to m.

Finally, write $m = \sum_{j \geq 0} \pi^j m_j$ again, and pick $N \in \mathbb{N}$. Since M is integrable, for every $0 \leq j < N$, $E_{\alpha_i}^{(r)} m_j = 0$ for $r >> 0$. So there exists $R > 0$ such that for all $r > R$ and for any $0 \leq j < N$, $E_{\alpha_i}^{(r)} m_j = 0$. Then we have

$$E_{\alpha_i}^{(r)} m = \sum_{j \geq N} \pi^j E_{\alpha_i}^{(r)} m_j \in \pi^N \widehat{M}$$

for $r > R$. So $E_{\alpha_i}^{(r)} m \to 0$ as $r \to \infty$ as required. □

Remark 5.1.6. The proof of Proposition 5.1.5(ii) was adapted for quantum groups from a proof that was communicated to us privately by Simon Wadsley.

We aim to show a converse to Proposition 5.1.5(i). Similarly to the uncompleted situation, it boils down to showing that closed submodules of topologically integrable modules are topologically integrable. We are only able to do this for torsion-free modules, but this is sufficient for our needs.

Definition 5.1.7. A Banach $\widehat{U^{\mathrm{res}}(\mathfrak{b})}_L$-module $\mathcal{M}$ is called *topologically integrable* if its unit ball $\mathcal{M}^\circ$ is a topologically integrable $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-module. A morphism of topologically integrable $\widehat{U^{\mathrm{res}}(\mathfrak{b})}_L$-modules is a continuous $\widehat{U^{\mathrm{res}}(\mathfrak{b})}_L$-linear map.

Note that a topologically integrable $\widehat{U^{\mathrm{res}}(\mathfrak{b})}_L$-module is by definition topologically $(\widehat{U^{\mathrm{res}}})^0_L$-semisimple in the sense of Definition 3.5.1.

Let $\widehat{\mathcal{O}_q(B)} := \widehat{\mathcal{B}_q} \otimes_R L$. By Corollary 3.1.5 this is a Banach Hopf algebra. We now use the above results and the formalism of topologically semisimple modules as in Definition 3.5.1 to obtain a description of the category of Banach $\widehat{\mathcal{O}_q(B)}$-comodules.

Proposition 5.1.8. *Let $\mathcal{M}$ be a π-torsion free $\widehat{\mathcal{B}_q}$-comodule. Then $\mathcal{M}$ is a topologically integrable $\widehat{U^{res}(\mathfrak{b})}$-module.*

Proof. We have the comodule map $\rho : \mathcal{M} \to \mathcal{M}\widehat{\otimes}_R\mathcal{B}_q$ which is a split injection. As it is split, we must have

$$\pi^n(\mathcal{M}\widehat{\otimes}_R\mathcal{B}_q) \cap \rho(\mathcal{M}) = \pi^n\rho(\mathcal{M}) = \rho(\pi^n\mathcal{M})$$

so that ρ is in fact an isometry with respect to the π-adic norms. Moreover ρ is a comodule homomorphism if we give $\mathcal{M}\widehat{\otimes}_R\mathcal{B}_q$ the comodule map $\mathrm{id}\,\widehat{\otimes}\Delta$ by Remark 2.1.14. Hence this gives rise to a $\widehat{U^{\mathrm{res}}(\mathfrak{b})}_L$-linear isometry $\mathcal{M}_L \to \mathcal{M}_L\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ by Proposition 3.1.4 and Corollary 5.1.3.

Note that $\mathcal{M}\widehat{\otimes}_R\mathcal{B}_q$ is the π-adic completion of $\mathcal{M} \otimes_R \mathcal{B}_q$, which is a $\mathcal{B}_q$-comodule via $\mathrm{id} \otimes \Delta$. Since $\mathcal{B}_q$-comodules are integrable $U^{\mathrm{res}}(\mathfrak{b})$-modules by Theorem 2.5.12, it follows from Proposition 5.1.5(ii) that $\mathcal{M}\widehat{\otimes}_R\mathcal{B}_q$ is topologically integrable, hence so is $\mathcal{M}_L\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$. Now we identify $\mathcal{M}$ with its image in $\mathcal{M}\widehat{\otimes}_R\mathcal{B}_q = (\mathcal{M}_L\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)})^\circ$. Since the map was an isometry we also have $\mathcal{M} = \mathcal{M}_L \cap \mathcal{M}\widehat{\otimes}_R\mathcal{B}_q$. Pick $m \in \mathcal{M}$. Then inside $\mathcal{M}_L\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ we automatically have $m = \sum_{\lambda\in P} m_\lambda$ and $E_{\alpha_i}^{(r)} m \to 0$ as $r \to \infty$. So we just need to check that each $m_\lambda \in \mathcal{M}$. However by Theorem 3.5.4, the m_λ are uniquely determined by m and must belong to $\mathcal{M}_L$ since it is complete hence closed in $\mathcal{M}_L\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$. On the other hand, since $\mathcal{M}\widehat{\otimes}_R\mathcal{B}_q$ is topologically integrable and $m \in \mathcal{M} \subset \mathcal{M}\widehat{\otimes}_R\mathcal{B}_q$ we must have $m_\lambda \in \mathcal{M}\widehat{\otimes}_R\mathcal{B}_q$ for all λ. Therefore each $m_\lambda \in \mathcal{M}\widehat{\otimes}_R\mathcal{B}_q \cap \mathcal{M}_L = \mathcal{M}$ as required. $\square$

Note that by Proposition 5.1.5(i) there is a canonical functor between the category of topologically integrable $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-modules and $\widehat{\mathcal{B}_q}$-comodules. Indeed, given a module map $f : \mathcal{M} \to \mathcal{N}$ its restriction modulo π^n is a module map between two integrable $U^{\mathrm{res}}(\mathfrak{b})$-modules by definition, hence is a comodule homomorphism. Passing to the inverse limit, f is a $\widehat{\mathcal{B}_q}$-comodule homomorphism.

Corollary 5.1.9. *The canonical functor between the category of topologically integrable $\widehat{U^{res}(\mathfrak{b})}$-modules and the category of $\widehat{\mathcal{B}_q}$-comodules restricts to an equivalence of categories between the full subcategories of π-torsion free objects.*

Proof. By Proposition 5.1.8, the restriction of the functor to the torsion-free modules is essentially surjective. It is evidently faithful. Moreover, it is full by Corollary 5.1.3. $\square$

If $\mathcal{M}$ is a topologically integrable $\widehat{U^{\mathrm{res}}(\mathfrak{b})_L}$-module then we may apply the above functor to its unit ball and extend scalars to construct a functor to $\mathbf{Comod}(\widehat{\mathcal{O}_q(B)})$. This will give our promised Theorem C from the Introduction. First we record the following:

Lemma 5.1.10. *Suppose that $\mathcal{M}$ is a Banach $\mathcal{O}_q(B)$-comodule, equipped with the gauge norm of its unit ball $\mathcal{M}^\circ$. Let $\mathcal{N}$ be a subcomodule, equipped with the subspace norm. If the comodule map $\rho : \mathcal{M} \to \mathcal{M}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ has norm ≤ 1, then ρ makes $\mathcal{N}^\circ$ into a $\widehat{\mathcal{B}_q}$-comodule.*

Proof. By Lemma 2.7.17, $\mathcal{N}$ is equipped with the gauge norm of its unit ball. Moreover, by definition, so is $\widehat{\mathcal{O}_q(B)}$. Thus, by Proposition 2.8.9(ii) and Proposition 2.7.8, the unit balls of $\mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ and $\mathcal{M}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ are $\mathcal{N}^\circ\widehat{\otimes}_R\mathcal{B}_q$ and $\mathcal{M}^\circ\widehat{\otimes}_R\mathcal{B}_q$ respectively. Furthermore, the canonical map $\mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)} \to \mathcal{M}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ is an isometry by Proposition 2.8.9(iii). Hence, since ρ has norm ≤ 1, so does its restriction to $\mathcal{N}$ and thus ρ maps $\mathcal{N}^\circ$ to $\mathcal{N}^\circ\widehat{\otimes}_R\mathcal{B}_q$. The rest now follows from the comodule axioms on ρ. $\square$

Proof of Theorem C. The functor is evidently faithful. If $f : \mathcal{M} \to \mathcal{N}$ is a comodule homomorphism between two topologically integrable $\widehat{U^{\mathrm{res}}(\mathfrak{b})_L}$-modules, viewed as $\widehat{\mathcal{O}_q(B)}$-comodules under our functor, we want to show that f is $\widehat{U^{\mathrm{res}}(\mathfrak{b})_L}$-linear. But there exists $n \geq 0$ such that $f(\mathcal{M}^\circ) \subseteq \pi^{-n}\mathcal{N}^\circ$, and by definition $\pi^{-n}\mathcal{N}^\circ$ is a topologically integrable $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-module. Hence it must be that f is $\widehat{U^{\mathrm{res}}(\mathfrak{b})_L}$-linear by Corollary 5.1.3. Thus we see that the functor is full, and we just need to show that it is essentially surjective. Now suppose that $\mathcal{N}$ is a Banach $\widehat{\mathcal{O}_q(B)}$-comodule. Then there is a split injection $\rho : \mathcal{N} \to \mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ which is therefore strict by Lemma 2.7.20(iii). Moreover ρ is a comodule homomorphism where we give the right hand side the comodule map $\mathrm{id}\widehat{\otimes}\widehat{\Delta}$ by Remark 2.1.14. Hence $\mathcal{N}$ is topologically isomorphic to a subcomodule $\mathcal{M}$ of $\mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$, where $\mathcal{M}$ is equipped with the subspace topology. Without loss of generality, we may assume that $\mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ is equipped with the gauge norm associated to $\mathcal{N}^\circ\widehat{\otimes}_R\mathcal{B}_q$. We note that since $\mathrm{id}\widehat{\otimes}\widehat{\Delta}$ has norm ≤ 1. Thus we see by Lemma 5.1.10 that $\mathcal{M}^\circ$ is a $\widehat{\mathcal{B}_q}$-comodule, and therefore is a topologically integrable $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-module by Proposition 5.1.8. So we have that $\mathcal{M}$ is in the image of our functor. $\square$

5.2 The integral analytic quantum flag variety

We now begin to define analytic analogues of our quantum flag varieties, first working with integral forms.

Given two $\widehat{\mathcal{B}_q}$-comodules $\mathcal{M}, \mathcal{N}$ we can define a tensor comodule structure on $\mathcal{M}\widehat{\otimes}_R\mathcal{N}$: there is an R-module map

$$\mathcal{M} \otimes_R \mathcal{B}_q \otimes_R \mathcal{N} \otimes_R \mathcal{B}_q \to \mathcal{M} \otimes_R \mathcal{N} \otimes_R \mathcal{B}_q$$

given by swapping the two middle terms first and then multiplying in $\mathcal{B}_q$. Pre-composing the π-adic completion of this map with $\rho_{\mathcal{M}}\widehat{\otimes}\rho_{\mathcal{N}}$ gives a map

$$\mathcal{M}\widehat{\otimes}_R\mathcal{N} \to \mathcal{M}\widehat{\otimes}_R\mathcal{N}\widehat{\otimes}_R\mathcal{B}_q$$

which is easily checked to satisfy the comodule axioms. Indeed since the counit and comultiplication on $\mathcal{B}_q$ are algebra homomorphisms it's enough to check that

$$\begin{aligned}(\rho_{\mathcal{M}}\widehat{\otimes}\,\mathrm{id}_{\mathcal{B}_q}\widehat{\otimes}\rho_{\mathcal{N}}\widehat{\otimes}\,\mathrm{id}_{\mathcal{B}_q}) \circ (\rho_{\mathcal{M}}\widehat{\otimes}\rho_{\mathcal{N}}) &= (\mathrm{id}_{\mathcal{M}}\widehat{\otimes}\Delta\widehat{\otimes}\,\mathrm{id}_{\mathcal{N}}\widehat{\otimes}\Delta) \circ (\rho_{\mathcal{M}}\widehat{\otimes}\rho_{\mathcal{N}}), \text{ and} \\ (\mathrm{id}_{\mathcal{M}}\widehat{\otimes}\varepsilon\widehat{\otimes}\,\mathrm{id}_{\mathcal{N}}\widehat{\otimes}\varepsilon) \circ (\rho_{\mathcal{M}}\widehat{\otimes}\rho_{\mathcal{N}}) &= \mathrm{id}_{\mathcal{M}\widehat{\otimes}_R\mathcal{N}}\end{aligned}$$

where in the latter we make the identification $\mathcal{M}\widehat{\otimes}_R R\widehat{\otimes}_R\mathcal{N}\widehat{\otimes}_R R \cong \mathcal{M}\widehat{\otimes}_R\mathcal{N}$. These both follow since $\mathcal{M}$ and $\mathcal{N}$ are comodules.

Definition 5.2.1. The category $\widehat{\mathscr{C}_R}$ denotes the category whose objects are π-adically complete $\widehat{\mathcal{A}_q}$-modules and $\widehat{\mathcal{B}_q}$-comodules $\mathcal{M}$ such that the action map $\widehat{\mathcal{A}_q}\widehat{\otimes}_R\mathcal{M} \to \mathcal{M}$ is a comodule homomorphism with $\widehat{\mathcal{A}_q}\widehat{\otimes}_R\mathcal{M}$ being given the tensor comodule described above. Morphisms in $\widehat{\mathscr{C}_R}$ are R-linear maps preserving both structures. We say that $\mathcal{M} \in \widehat{\mathscr{C}_R}$ is *coherent* if it is finitely generated over $\widehat{\mathcal{A}_q}$ and denote the full subcategory of coherent modules by $\mathrm{coh}(\widehat{\mathscr{C}_R})$.

Given $\mathcal{M} \in \widehat{\mathscr{C}_R}$, its *global sections* are defined to be $\Gamma(\mathcal{M}) := \mathrm{Hom}_{\widehat{\mathscr{C}_R}}(\widehat{\mathcal{A}_q}, \mathcal{M})$. The map $\Gamma(\mathcal{M}) \to \mathcal{M}$ defined by $f \mapsto f(1)$ gives an isomorphism between $\Gamma(\mathcal{M})$ and the R-module $\{m \in \mathcal{M} : \rho_{\mathcal{M}}(m) = m \otimes 1\}$.

Lemma 5.2.2. *Suppose $\mathcal{M} \in \widehat{\mathscr{C}_R}$. Then $\mathcal{M}/\pi^n\mathcal{M} \in \mathscr{C}_R$ for all $n \geq 0$ and is coherent if $\mathcal{M}$ was coherent.*

Proof. Clearly $\mathcal{M}/\pi^n\mathcal{M}$ is an $\widehat{\mathcal{A}_q}/\pi^n\widehat{\mathcal{A}_q} \cong \mathcal{A}_q/\pi^n\mathcal{A}_q$-module and so an $\mathcal{A}_q$-module, and it is finitely generated if $\mathcal{M}$ was finitely generated. Moreover we know that $\mathcal{M}/\pi^n\mathcal{M}$ is a $\mathcal{B}_q$-comodule by Lemma 5.1.2. So we just need to show that the $\mathcal{A}_q$-action map is a comodule homomorphism, i.e. that for all $n \geq 1$, the diagram

$$\begin{tikzcd} \mathcal{A}_q \otimes_R \mathcal{M}/\pi^n\mathcal{M} \arrow[r] \arrow[d] & \mathcal{M}/\pi^n\mathcal{M} \arrow[d, "\rho_n"] \\ \mathcal{A}_q \otimes_R \mathcal{M}/\pi^n\mathcal{M} \otimes_R \mathcal{B}_q \arrow[r] & \mathcal{M}/\pi^n\mathcal{M} \otimes_R \mathcal{B}_q \end{tikzcd}$$

commutes. But this follows by tensoring with $R/\pi^n R$ the diagram

$$\begin{tikzcd} \mathcal{A}_q\widehat{\otimes}_R\mathcal{M} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "\rho"] \\ \mathcal{A}_q\widehat{\otimes}_R\mathcal{M}\widehat{\otimes}_R\mathcal{B}_q \arrow[r] & \mathcal{M}\widehat{\otimes}_R\mathcal{B}_q \end{tikzcd}$$

which commutes since $\mathcal{M} \in \widehat{\mathscr{C}_R}$. □

We will now see how to construct objects in $\widehat{\mathscr{C}_R}$ from those in $\mathscr{C}_R$.

Lemma 5.2.3. *The assignment $M \mapsto \widehat{M}$ defines a functor $\mathscr{C}_R \to \widehat{\mathscr{C}_R}$.*

Proof. Suppose $M \in \mathscr{C}_R$ and let $\rho : M \to M \otimes_R \mathcal{B}_q$ denote the comodule map of M. Then clearly $\pi^n M$ is an integrable $U^{\mathrm{res}}(\mathfrak{b})$-submodule, hence so is the quotient $M/\pi^n M$, giving comodule maps $\rho_n : M/\pi^n M \to M/\pi^n M \otimes_R \mathcal{B}_q$ for every $n \geq 1$. So we can define

$$\widehat{\rho} = \varprojlim \rho_n : \widehat{M} \to \varprojlim (M/\pi^n M \otimes_R \mathcal{B}_q) \cong M \widehat{\otimes}_R \mathcal{B}_q,$$

which is just the π-adic completion of ρ. This gives the structure of a $\widehat{\mathcal{B}_q}$-comodule since all the maps ρ_n are comodule maps. We also have that $\widehat{M}$ is an $\widehat{\mathcal{A}_q}$-module and the action map is a comodule homomorphism since the $\mathcal{A}_q$-action map on $M/\pi^n M$ is a comodule homomorphism for every $n \geq 1$.

Finally, given a morphism $f : M \to N$ in $\mathscr{C}_R$, we have that the induced map $\widehat{f} : \widehat{M} \to \widehat{N}$ is an $\widehat{\mathcal{A}_q}$-module map. Moreover, by functoriality of taking π-adic completion in R-modules, the diagram

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\widehat{f}} & \widehat{N} \\ \Big\downarrow{\scriptstyle \widehat{\rho_M}} & & \Big\downarrow{\scriptstyle \widehat{\rho_N}} \\ M \widehat{\otimes}_R \mathcal{B}_q & \xrightarrow{f \widehat{\otimes} 1} & N \widehat{\otimes}_R \mathcal{B}_q \end{array}$$

commutes as required. □

Proposition 5.2.4. *If $\mathcal{M}, \mathcal{N} \in \widehat{\mathscr{C}_R}$ then there is a canonical isomorphism*

$$\mathrm{Hom}_{\widehat{\mathscr{C}_R}}(\mathcal{M}, \mathcal{N}) \cong \varprojlim \mathrm{Hom}_{\mathscr{C}_R}(\mathcal{M}/\pi^n \mathcal{M}, \mathcal{N}/\pi^n \mathcal{N}).$$

In particular we have $\Gamma(\mathcal{N}) \cong \varprojlim \Gamma(\mathcal{N}/\pi^n \mathcal{N})$ and so $\Gamma(\widehat{M}) \cong \varprojlim \Gamma(M/\pi^n M)$ for $M \in \mathscr{C}_R$.

Proof. Any $f : \mathcal{M} \to \mathcal{N}$ in $\widehat{\mathscr{C}_R}$ induces maps $f_n : \mathcal{M}/\pi^n \mathcal{M} \to \mathcal{N}/\pi^n \mathcal{N}$ in $\mathscr{C}_R$ for all n, and these determine f uniquely by passing to the inverse limit. Hence there is a canonical injection

$$\mathrm{Hom}_{\widehat{\mathscr{C}_R}}(\mathcal{M}, \mathcal{N}) \to \varprojlim \mathrm{Hom}_{\mathscr{C}_R}(\mathcal{M}/\pi^n \mathcal{M}, \mathcal{N}/\pi^n \mathcal{N})$$

which is surjective by the same argument as in Corollary 5.1.3. Hence by putting $\mathcal{M} = \widehat{\mathcal{A}_q}$ and using the fact that

$$\mathrm{Hom}_{\mathscr{C}_R}(\mathcal{A}_q/\pi^n \mathcal{A}_q, \mathcal{N}/\pi^n \mathcal{N}) \cong \mathrm{Hom}_{\mathscr{C}_R}(\mathcal{A}_q, \mathcal{N}/\pi^n \mathcal{N}) = \Gamma(\mathcal{N}/\pi^n \mathcal{N}),$$

we get the result on global sections. □

Remark 5.2.5. This result is something one would expect given what the sections of an inverse limit of sheaves on a topological space are, see [41, Proposition II.9.2].

Recall the Ore localisations $\mathcal{A}_{q,w}$ of $\mathcal{A}_q$. We had a category $\mathscr{C}_R^w$ of $\mathcal{B}_q$-equivariant $\mathcal{A}_{q,w}$-modules. We may analogously define categories $\widehat{\mathscr{C}_R^w}$ as follows. The objects are π-adically complete $\widehat{\mathcal{A}_{q,w}}$-modules and $\widehat{\mathcal{B}_q}$-comodules $\mathcal{M}$ such that the action map $\widehat{\mathcal{A}_{q,w}} \widehat{\otimes}_R \mathcal{M} \to \mathcal{M}$ is a comodule homomorphism with $\widehat{\mathcal{A}_{q,w}} \widehat{\otimes}_R \mathcal{M}$ being given the tensor comodule structure. Morphisms in $\widehat{\mathscr{C}_R^w}$ are R-linear maps preserving both structures. The analogues of Lemma 5.2.2, Lemma 5.2.3 and Proposition 5.2.4 apply to $\widehat{\mathscr{C}_R^w}$ as well with identical proofs.

Now, the localisation map $\varphi : \mathcal{A}_q \to \mathcal{A}_{q,w}$ gives rise to a map $\widehat{\varphi} : \widehat{\mathcal{A}_q} \to \widehat{\mathcal{A}_{q,w}}$, and recall that $\widehat{\mathcal{A}_q}$ is Noetherian by Corollary 2.6.3.

Lemma 5.2.6. *Let $\mathcal{M}$ be an $\widehat{\mathcal{A}_q}$-module and let $n \geq 1$. We have:*

(i) $\widehat{\mathcal{A}_{q,w}}$ is a Noetherian R-algebra, and is flat as an $\widehat{\mathcal{A}_q}$-module; and

(ii) there is an isomorphism

$$\left(\widehat{\mathcal{A}_{q,w}} \otimes_{\widehat{\mathcal{A}_q}} \mathcal{M}\right) / \pi^n \left(\widehat{\mathcal{A}_{q,w}} \otimes_{\widehat{\mathcal{A}_q}} \mathcal{M}\right) \cong \mathcal{A}_{q,w} \otimes_{\mathcal{A}_q} (\mathcal{M}/\pi^n \mathcal{M})$$

of $\mathcal{A}_{q,w}$-modules.

Proof. (i) That $\widehat{\mathcal{A}_{q,w}}$ is Noetherian will follow from Proposition 2.6.2(i) if we show that $\mathcal{A}_{q,w}/\pi\mathcal{A}_{q,w}$ is Noetherian. But, by Proposition 2.5.18(iv), it is the localisation of the commutative Noetherian ring $\mathcal{A}_q/\pi\mathcal{A}_q$ at the image of S_w in the quotient, hence it is Noetherian (see [34, Corollary 2.3]). Then by Lemma 2.6.4, the flatness will follow if we show that the maps

$$\varphi_a : \mathcal{A}_q/\pi^a\mathcal{A}_q \to \mathcal{A}_{q,w}/\pi^a\mathcal{A}_{q,w}$$

are all flat for $a \geq 1$. But we have more generally that $\mathcal{A}_{q,w}/\pi^a\mathcal{A}_{q,w}$ is the localisation of $\mathcal{A}_q/\pi^a\mathcal{A}_q$ at the image of S_w in the quotient, and by [63, Proposition 2.1.16] localisation is flat. So the maps φ_a are flat.

(ii) We have

$$\begin{aligned}
\left(\widehat{\mathcal{A}_{q,w}} \otimes_{\widehat{\mathcal{A}_q}} \mathcal{M}\right) / \pi^n \left(\widehat{\mathcal{A}_{q,w}} \otimes_{\widehat{\mathcal{A}_q}} \mathcal{M}\right) &\cong \left(\widehat{\mathcal{A}_{q,w}}/\pi^n\widehat{\mathcal{A}_{q,w}}\right) \otimes_{\widehat{\mathcal{A}_{q,w}}} \left(\widehat{\mathcal{A}_{q,w}} \otimes_{\widehat{\mathcal{A}_q}} \mathcal{M}\right) \\
&\cong \left(\widehat{\mathcal{A}_{q,w}}/\pi^n\widehat{\mathcal{A}_{q,w}}\right) \otimes_{\widehat{\mathcal{A}_q}} \mathcal{M} \\
&\cong \left(\mathcal{A}_{q,w}/\pi^n\mathcal{A}_{q,w}\right) \otimes_{\widehat{\mathcal{A}_q}} \mathcal{M} \\
&\cong \left(\mathcal{A}_{q,w} \otimes_{\mathcal{A}_q} \mathcal{A}_q/\pi^n\mathcal{A}_q\right) \otimes_{\widehat{\mathcal{A}_q}} \mathcal{M} \\
&\cong \mathcal{A}_{q,w} \otimes_{\mathcal{A}_q} \left(\widehat{\mathcal{A}_q}/\pi^n\widehat{\mathcal{A}_q} \otimes_{\widehat{\mathcal{A}_q}} \mathcal{M}\right) \\
&\cong \mathcal{A}_{q,w} \otimes_{\mathcal{A}_q} (\mathcal{M}/\pi^n\mathcal{M})
\end{aligned}$$

as required. $\square$

Remark 5.2.7. We believe $\widehat{\mathcal{A}_{q,w}}$ to be a microlocalisation of $\widehat{\mathcal{A}_q}$ in the sense of [6, Section 2.4], but we haven't worked out the details.

Definition 5.2.8. Given an $\widehat{\mathcal{A}_q}$-module $\mathcal{M}$ and $w \in W$, we define the *w-localisation* of $\mathcal{M}$ to be the π-adic completion $\widehat{\mathcal{A}_{q,w}}\widehat{\otimes}_{\widehat{\mathcal{A}_q}}\mathcal{M}$ of $\widehat{\mathcal{A}_{q,w}} \otimes_{\widehat{\mathcal{A}_q}} \mathcal{M}$. We sometimes denote it as $S_w^{-1}\mathcal{M}$ by abuse of notation.

Proposition 5.2.9. *The functor $\widehat{f_w}^* : \mathcal{M} \mapsto \widehat{\mathcal{A}_{q,w}}\widehat{\otimes}_{\widehat{\mathcal{A}_q}}\mathcal{M}$ sends $\widehat{\mathscr{C}_R}$ to $\widehat{\mathscr{C}_R^w}$.*

Proof. By Lemma 5.2.6(ii), for any $\mathcal{M} \in \widehat{\mathscr{C}_R}$ and any $n \geq 1$ we have that

$$\left(\widehat{\mathcal{A}_{q,w}} \otimes_{\widehat{\mathcal{A}_q}} \mathcal{M}\right) / \pi^n \left(\widehat{\mathcal{A}_{q,w}} \otimes_{\widehat{\mathcal{A}_q}} \mathcal{M}\right) \cong S_w^{-1}(\mathcal{M}/\pi^n\mathcal{M}).$$

Now for each $n \geq 1$, $\mathcal{M}/\pi^n\mathcal{M} \in \mathscr{C}_R$ by Lemma 5.2.2, and so its localisation is in $\mathscr{C}_R^w$. The comodule maps ρ_n on $\mathcal{M}/\pi^n\mathcal{M}$ are compatible with the reduction maps and hence so are their localisations $S_w^{-1}\rho_n$. Thus we may define a comodule map

$$S_w^{-1}\rho = \varprojlim S_w^{-1}\rho_n : S_w^{-1}\mathcal{M} \to S_w^{-1}\mathcal{M}\widehat{\otimes}_R\mathcal{B}_q$$

and the $\widehat{\mathcal{A}_{q,w}}$-action is a comodule homomorphism since it is modulo π^n for all n.

Finally $\widehat{f_w}^*$ really is a functor since it sends a given morphism $\varphi : \mathcal{M} \to \mathcal{N}$ to $\varprojlim S_w^{-1}\varphi_n$ where $\varphi_n : \mathcal{M}/\pi^n\mathcal{M} \to \mathcal{N}/\pi^n\mathcal{N}$, and each $S_w^{-1}\varphi_n$ is a morphism in $\mathscr{C}_R^w$. □

For each $i \in J$ we have a forgetful functor $(\widehat{f_{w_i}})_* : \widehat{\mathscr{C}_R^{w_i}} \to \widehat{\mathscr{C}_R}$ which is a right adjoint to $\widehat{f_{w_i}}^*$: this is easily seen from Proposition 5.2.4 since their restrictions modulo π^n are adjoints for all n. Let $\sigma_i := (\widehat{f_{w_i}})_* \circ \widehat{f_{w_i}}^*$ and, for any $\mathbf{i} = (i_1, \ldots, i_n) \in J^n$, set $\sigma_{\mathbf{i}} = \sigma_{i_1} \circ \cdots \circ \sigma_{i_n}$. We may then completely analogously as in (4.3) write down a complex

$$C^{\text{aug}} : \mathrm{id}_{\mathscr{C}_R} \to \bigoplus_{i\in J} \sigma_i \to \bigoplus_{\mathbf{i}\in J^2} \sigma_{\mathbf{i}} \to \bigoplus_{\mathbf{i}\in J^3} \sigma_{\mathbf{i}} \to \cdots$$

which we may post-compose with the functor of taking global sections to obtain a complex $\check{C}^{\text{aug}}$, which we still call the *augmented standard complex* of Γ. We often consider those complexes without the left hand most term, which we denote by C and $\check{C}$. The complex $\check{C}$ is called the *standard complex*.

Remark 5.2.10. Note that by Lemma 5.2.6(ii) and Proposition 5.2.4, for any $\mathcal{M} \in \widehat{\mathscr{C}_R}$, we have isomorphisms

$$C(\mathcal{M})^{\text{aug}} \cong \varprojlim C^{\text{aug}}(\mathcal{M}/\pi^n\mathcal{M}) \quad \text{and} \quad \check{C}(\mathcal{M})^{\text{aug}} \cong \varprojlim \check{C}^{\text{aug}}(\mathcal{M}/\pi^n\mathcal{M}),$$

and the same is true for the non-augmented complexes.

Definition 5.2.11. Given $\mathcal{M} \in \widehat{\mathscr{C}_R}$ and $i \geq 0$, we define the *Čech cohomology* $\check{H}^i(\mathcal{M})$ to be the i-th cohomology group of $\check{C}(\mathcal{M})$.

Thus we are considering the cohomology of a complex of abelian groups which is given as an inverse limit of complexes. To such a situation, there is a general result telling us how to relate the cohomology groups of an inverse limit of complexes to the inverse limit of the cohomology groups:

Proposition 5.2.12 ([40, Proposition 0.13.2.3]). *Let $(C_a^\bullet)_{a\geq 1}$ be a projective system of complexes of abelian groups $C_a^\bullet = (C_a^m)_{m\in\mathbb{Z}}$. For each $n \in \mathbb{Z}$, there is a canonical functorial homomorphism*

$$h_n : H^n(\varprojlim C_a^\bullet) \to \varprojlim H^n(C_a^\bullet).$$

If, for all $m \in \mathbb{Z}$, the projective system $(C_a^m)_{a\geq 1}$ satisfies the Mittag-Leffler condition, then h_n is surjective for all $n \in \mathbb{Z}$. If furthermore, for a given $n \in \mathbb{Z}$, the projective system $(H^{n-1}(C_a^\bullet))_{a\geq 1}$ satisfies the Mittag-Leffler condition, then the map h_n is an isomorphism.

5.3 The analytic quantum flag variety

Recall the Banach Hopf algebras $\widehat{\mathcal{O}_q}$ and $\widehat{\mathcal{O}_q(B)}$, and that $\widehat{\mathcal{O}_q}$ is Noetherian by Corollary 2.6.3 and Lemma 2.6.1. Note that, completely analogously to the discussion preceeding Definition 5.2.1, given two Banach $\widehat{\mathcal{O}_q(B)}$-comodules $\mathcal{M}$ and $\mathcal{N}$, we may define a tensor comodule structure on $\mathcal{M}\widehat{\otimes}_L\mathcal{N}$.

Definition 5.3.1. We let $\widehat{\mathcal{M}_{B_q}(G_q)}$ denote the category whose objects are triples $(\mathcal{M},\alpha,\beta)$ where $\mathcal{M}$ is an L-Banach space, $\alpha : \widehat{\mathcal{O}_q}\widehat{\otimes}_L\mathcal{M} \to \mathcal{M}$ is a left $\widehat{\mathcal{O}_q}$-module action and $\beta : \mathcal{M} \to \mathcal{M}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ is a right $\widehat{\mathcal{O}_q(B)}$-comodule action, such that α is a comodule homomorphism where $\widehat{\mathcal{O}_q}\widehat{\otimes}_L\mathcal{M}$ is given the tensor comodule structure. The morphisms are just the continuous linear maps which are both module and comodule homomorphisms.

A triple $(\mathcal{M},\alpha,\beta)$ where $\mathcal{M}$ is a finitely generated $\widehat{\mathcal{O}_q}$-module is called *coherent*, and we let $\mathrm{coh}(\widehat{\mathcal{M}_{B_q}(G_q)})$ denote the full subcategory of coherent modules.

Of course, $\widehat{\mathcal{O}_q} \in \widehat{\mathcal{M}_{B_q}(G_q)}$ and is coherent by definition. By Lemma 2.8.6, the full subcategory of finitely generated modules in $\mathbf{Mod}(\widehat{\mathcal{O}_q})$ is abelian. Similarly $\mathrm{coh}(\widehat{\mathcal{M}_{B_q}(G_q)})$ is also abelian. However there is no guarantee that it has enough injectives. Instead we will work in $\widehat{\mathcal{M}_{B_q}(G_q)}$.

We begin by generalising one of the adjunction from Section 4.2. There is an adjoint pair of functors (θ^*,θ_*) between $\mathbf{Mod}(\widehat{\mathcal{O}_q})$ and $\widehat{\mathcal{M}_{B_q}(G_q)}$. The functor $\theta_* : \mathbf{Mod}(\widehat{\mathcal{O}_q}) \to \widehat{\mathcal{M}_{B_q}(G_q)}$ is given by $\mathcal{N} \mapsto \mathcal{N}\widehat{\otimes}_L\mathcal{O}_q(B)$ where $\widehat{\mathcal{O}_q}$ acts on $\theta_*(\mathcal{N})$ via the tensor action and the $\widehat{\mathcal{O}_q(B)}$-coaction comes from the second factor, while $\theta^* : \widehat{\mathcal{M}_{B_q}(G_q)} \to \mathbf{Mod}(\widehat{\mathcal{O}_q})$ is just the forgetful functor. The bijection making this an adjunction is given as follows. Let $\mathcal{M} \in \widehat{\mathcal{M}_{B_q}(G_q)}$ and $\mathcal{N} \in \mathbf{Mod}(\widehat{\mathcal{O}_q})$, and let $\rho : \mathcal{M} \to \mathcal{M}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ and $\varepsilon : \widehat{\mathcal{O}_q(B)} \to L$ be the comodule map and the counit of $\widehat{\mathcal{O}_q(B)}$ respectively. Given a module homomorphism $f : \mathcal{M} \to \mathcal{N}$, construct a morphism $g : \mathcal{M} \to \mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ in $\widehat{\mathcal{M}_{B_q}(G_q)}$ by taking the composite $(f\widehat{\otimes}\,\mathrm{id}) \circ \rho$. Conversely, given a morphism $g : \mathcal{M} \to \mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ in $\widehat{\mathcal{M}_{B_q}(G_q)}$, we construct a module homomorphism $f : \mathcal{M} \to \mathcal{N}$ by taking the composite $(\mathrm{id}\widehat{\otimes}\varepsilon) \circ g$. Having established this, we can now prove:

Lemma 5.3.2. *The category $\widehat{\mathcal{M}_{B_q}(G_q)}$ is quasi-abelian and has enough injectives.*

Proof. The proof is entirely analogous to the proofs of Lemma 2.9.4 and Proposition 2.9.10, using the adjunction (θ^*,θ_*). □

We now define a global section functor. First note that for any $\mathcal{M},\mathcal{N} \in \widehat{\mathcal{M}_{B_q}(G_q)}$, the space $\mathrm{Hom}_{\widehat{\mathcal{M}_{B_q}(G_q)}}(\mathcal{M},\mathcal{N})$ is a closed subspace of the Banach space $\mathrm{Hom}_L(\mathcal{M},\mathcal{N})$ of all bounded linear maps from $\mathcal{M}$ to $\mathcal{N}$. In particular it is itself a Banach space.

Definition 5.3.3. The global section functor $\Gamma : \widehat{\mathcal{M}_{B_q}(G_q)} \to \mathbf{Ban}_L$ is defined to be the functor

$$\Gamma : \mathcal{M} \mapsto \mathrm{Hom}_{\widehat{\mathcal{M}_{B_q}(G_q)}}(\widehat{\mathcal{O}_q},\mathcal{M}).$$

Alternatively, let $\mathcal{M}^{B_q} := \{m \in \mathcal{M} : \rho(m) = m \otimes 1\}$, a closed subspace of $\mathcal{M}$. We may think of global sections as taking B_q-invariants since there is a canonical topological isomorphism

$$\begin{aligned} \Gamma(\mathcal{M}) &\xrightarrow{\sim} \mathcal{M}^{B_q} \\ f &\longmapsto f(1) \end{aligned}$$

by Corollary 2.7.12.

Since Γ is just a hom functor, and since strict exact sequences of Banach spaces are just algebraically exact sequences by Example 2.9.6, it follows that Γ is left exact. In fact it clearly satisfies the conditions of Proposition 2.9.13(iv). Hence, as $\widehat{\mathcal{M}_{B_q}(G_q)}$ has enough injectives, this functor is right derivable and extends to a left exact functor $\mathcal{LH}(\widehat{\mathcal{M}_{B_q}(G_q)}) \to \mathcal{LH}(\mathbf{Ban}_L)$ with the same cohomology, and we denote by $R^i\Gamma$ the corresponding cohomology groups in $\mathcal{LH}(\mathbf{Ban}_L)$.

The functor $\mathcal{M} \mapsto \mathcal{M}_L$ of course sends $\widehat{\mathscr{C}_R}$ to $\widehat{\mathcal{M}_{B_q}(G_q)}$. We show that this functor is in fact essentially surjective. In the next proof, we will use the fact that the unit ball of a completed tensor product $\mathcal{M}\widehat{\otimes}_L\mathcal{N}$ of Banach spaces equipped with the gauge norms of their unit balls is $\mathcal{M}^\circ\widehat{\otimes}_R\mathcal{N}^\circ$ by Proposition 2.8.9(ii) and Proposition 2.7.8.

Proposition 5.3.4. *Let $\mathcal{M} \in \widehat{\mathscr{C}_R}$ and $\mathcal{N} \in \widehat{\mathcal{M}_{B_q}(G_q)}$. Then:*

(i) $\Gamma(\mathcal{M}_L) \cong \Gamma(\mathcal{M}) \otimes_R L$*; and*

(ii) *$\mathcal{N}$ contains an R-lattice $F_0\mathcal{N}$ which is an element of $\widehat{\mathscr{C}_R}$. Moreover, $F_0\mathcal{N}$ can be chosen to be coherent if $\mathcal{N} \in \operatorname{coh}(\widehat{\mathcal{M}_{B_q}(G_q)})$.*

Proof. The proof of (i) is the same as the first part of the proof of Proposition 4.3.10. For (ii), notice that the adjunction morphism $\mathcal{N} \to \theta_*\theta^*(\mathcal{N}) = \mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ is just the comodule map ρ, which has a left inverse given by $1\widehat{\otimes}\varepsilon$, so which is a strict injection by Lemma 2.7.20(iii). By definition of $\widehat{\mathcal{M}_{B_q}(G_q)}$, the map ρ is $\widehat{\mathcal{O}_q}$-linear if we give $\mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ the tensor module structure. Thus, by Remark 2.1.14, we see that $\mathcal{N}$ is isomorphic to a subobject of $\mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ in $\widehat{\mathcal{M}_{B_q}(G_q)}$, where $\mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ has the tensor $\widehat{\mathcal{O}_q}$-module structure and comodule structure given by $\mathrm{id}\,\widehat{\otimes}\widehat{\Delta}$. By initially equipping $\mathcal{N}$ with the gauge norm associtated with its unit ball, we may assume that $\mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ is equipped with the gauge norm associated to $\mathcal{N}^\circ\widehat{\otimes}_R\widehat{\mathcal{B}_q}$. From now on we identify $\mathcal{N}$ with its image in $\mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ via ρ, and in particular equipped with the subspace norm.

Let $F_0\mathcal{N} = \mathcal{N} \cap \mathcal{N}^\circ\widehat{\otimes}_R\widehat{\mathcal{B}_q}$ denote the unit ball of $\mathcal{N}$ with respect to that new norm. Note that the comodule map on $\mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ has norm at most 1, hence we see that $F_0\mathcal{N}$ is a $\widehat{\mathcal{B}_q}$-comodule by Lemma 5.1.10. Moreover, since the $\widehat{\mathcal{O}_q}$-action on $\mathcal{N}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ has norm at most 1, i.e. $\widehat{\mathcal{A}_q}$ preserves the unit ball $\mathcal{N}^\circ\widehat{\otimes}_R\widehat{\mathcal{B}_q}$, it follows that $F_0\mathcal{N}$ is an $\widehat{\mathcal{A}_q}$-module. Hence $F_0\mathcal{N}$ belongs to $\widehat{\mathscr{C}_R}$.

For the last part it is just left to check that it is finitely generated if $\mathcal{N} \in \operatorname{coh}(\widehat{\mathcal{M}_{B_q}(G_q)})$. By Lemma 2.8.6, the topology on $\mathcal{N}$ is the quotient topology from a surjection $(\widehat{\mathcal{O}_q})^a \twoheadrightarrow \mathcal{N}$ for some choice of generators in $\mathcal{N}$, and the unit ball $\mathcal{N}^\circ$ is given by the image of $(\widehat{\mathcal{A}_q})^a$. Since $\widehat{\mathcal{A}_q}$ is Noetherian, it follows from Corollary 2.8.7 that $F_0\mathcal{N}$ is finitely generated as required. $\square$

We now start working towards a Čech complex for computing the cohomology of elements of $\widehat{\mathcal{M}_{B_q}(G_q)}$. For each $w \in W$, let $\widehat{\mathcal{O}_{q,w}} := \widehat{\mathcal{A}_{q,w}} \otimes_R L$ and let $\widehat{\mathcal{M}_{B_q}(G_q)}_w$ denote the category of B-equivariant $\widehat{\mathcal{O}_{q,w}}$-modules. Specifically, the objects are triples $(\mathcal{M}, \alpha, \beta)$ where $\mathcal{M}$ is an L-Banach space, $\alpha : \widehat{\mathcal{O}_{q,w}} \widehat{\otimes}_L \mathcal{M} \to \mathcal{M}$ is a left $\widehat{\mathcal{O}_{q,w}}$-module action and $\beta : \mathcal{M} \to \mathcal{M} \widehat{\otimes}_L \widehat{\mathcal{O}_q(B)}$ is a right $\widehat{\mathcal{O}_q(B)}$-comodule action, such that α is a comodule homomorphism where $\widehat{\mathcal{O}_{q,w}} \widehat{\otimes}_L \mathcal{M}$ is given the tensor comodule structure. The morphisms are just the continuous linear maps which are both module and comodule homomorphisms.

Now recall that given continuous algebra homomorphism $f : A \to B$ between Banach algebras, there is an induced functor $f^* : \mathbf{Mod}(A) \to \mathbf{Mod}(B)$ given by $\mathcal{M} \mapsto B \widehat{\otimes}_A \mathcal{M}$. This functor has a right adjoint given by restriction. We now investigate when the functor f^* is strict exact. What we need is the following result:

Proposition 5.3.5 ([19, Proposition 1.3]). *Suppose A and B are Banach algebras, with unit balls A° and B° respectively. Furthermore, suppose that A° and B° are Noetherian R-algebras and assume that there is a continuous algebra homomorphism $f : A \to B$ such that $f(A^\circ) \subseteq B^\circ$. If f makes B° into a flat A°-module, then the functor f^* is strict exact.*

Now there is a continuous algebra homomorphism $\widehat{\mathcal{O}_q} \to \widehat{\mathcal{O}_{q,w}}$ induced from the localisation and this induces a functor $\widehat{f_w}^* : \mathbf{Mod}(\widehat{\mathcal{O}_q}) \to \mathbf{Mod}(\widehat{\mathcal{O}_{q,w}})$ with right adjoint $(\widehat{f_w})_*$ given by restriction as above.

Corollary 5.3.6. *Let $w \in W$. Then:*

(i) for any π-adically complete, π-torsion free $\widehat{\mathcal{A}_q}$-module $\mathcal{N}$,

$$(\widehat{\mathcal{A}_{q,w}} \widehat{\otimes}_{\widehat{\mathcal{A}_q}} \mathcal{N})_L \cong \widehat{\mathcal{O}_{q,w}} \widehat{\otimes}_{\widehat{\mathcal{O}_q}} \mathcal{N}_L.$$

In particular, if $\mathcal{N} \in \widehat{\mathscr{C}_R}$, then $\widehat{f_w}^*(\mathcal{N}_L) \cong \widehat{f_w}^*(\mathcal{N}) \otimes_R L$.

(ii) the functor $\widehat{f_w}^*$ defined above is strict exact and sends $\widehat{\mathcal{M}_{B_q}(G_q)}$ to $\widehat{\mathcal{M}_{B_q}(G_q)}_w$.

Proof. Part (i) follows from Lemma 5.2.6(i) by putting $A = \widehat{\mathcal{O}_q}$ and $B = \widehat{\mathcal{O}_{q,w}}$ in Corollary 2.8.15.

For (ii), it follows immediately from Proposition 5.3.5 that $\widehat{f_w}^*$ is strict exact by Lemma 5.2.6(i). Now by Proposition 5.3.4, given any $\mathcal{M} \in \widehat{\mathcal{M}_{B_q}(G_q)}$ we may assume that $\mathcal{M} = \mathcal{N} \otimes_R L$ for some $\mathcal{N} \in \widehat{\mathscr{C}_R}$. The result therefore follows from (i) and from Proposition 5.2.9. □

We now investigate exactness properties of global sections on the various localisations $\widehat{\mathcal{M}_{B_q}(G_q)}_w$. Recall that there is a comodule map $\Delta_w : \mathcal{A}_{q,w} \to \mathcal{A}_{q,w} \otimes_R \mathcal{B}_q$ which makes $\mathcal{A}_{q,w} \otimes_R \mathcal{B}_q$ into an $\mathcal{A}_{q,w}$-module. After taking π-adic completion and extending scalars, there is an analgous comodule map $\widehat{\Delta_w} : \widehat{\mathcal{O}_{q,w}} \to \widehat{\mathcal{O}_{q,w}} \widehat{\otimes}_L \widehat{\mathcal{O}_q(B)}$. Note then that given any Banach $\widehat{\mathcal{O}_{q,w}}$-module $\mathcal{M}$, we may give an $\widehat{\mathcal{O}_{q,w}}$-module structure to $\mathcal{M} \widehat{\otimes}_L \widehat{\mathcal{O}_q(B)}$ via the map $\widehat{\Delta_w}$, and an $\widehat{\mathcal{O}_q(B)}$-comodule structure by coacting on the right factor. Thus $\mathcal{M} \mapsto \mathcal{M} \widehat{\otimes}_L \widehat{\mathcal{O}_q(B)}$ defines a functor $\mathbf{Mod}(\widehat{\mathcal{O}_{q,w}}) \to \widehat{\mathcal{M}_{B_q}(G_q)}_w$, and this has a left adjoint given by the forgetful functor. With the same proofs as in Lemma 5.3.2 and Proposition 5.3.4, we obtain:

Lemma 5.3.7. *Let $w \in W$. Then:*

(i) the category $\widehat{\mathcal{M}_{B_q}(G_q)}_w$ is quasi-abelian and has enough injectives; and

(ii) for any $\mathcal{M} \in \widehat{\mathcal{M}_{B_q}(G_q)}_w$, there exists $F_0\mathcal{M} \in \widehat{\mathscr{C}_R^w}$ such that $\mathcal{M} \cong (F_0\mathcal{M})_L$.

Now we define the global section functor on $\widehat{\mathcal{M}_{B_q}(G_q)}_w$ to be the composite $\Gamma \circ (\widehat{f_w})_*$. We now need the following:

Lemma 5.3.8. *Suppose that $\mathcal{M} \in \widehat{\mathscr{C}_R}$ is π-torsion free and that $\mathcal{M}/\pi\mathcal{M}$ is Γ-acyclic. Then for every $n \geq 1$, the natural map*

$$\Gamma(\mathcal{M}/\pi^{n+1}\mathcal{M}) \to \Gamma(\mathcal{M}/\pi^n\mathcal{M})$$

is surjective. In particular, $(\Gamma(\mathcal{M}/\pi^n\mathcal{M}))_{n\geq 1}$ satisfies the Mittag-Leffler condition.

Proof. Since $\mathcal{M}$ has no π-torsion, we have a short exact sequence

$$0 \to \mathcal{M}/\pi\mathcal{M} \xrightarrow{\pi^n} \mathcal{M}/\pi^{n+1}\mathcal{M} \to \mathcal{M}/\pi^n\mathcal{M} \to 0$$

in $\mathscr{C}_R$ for each $n \geq 1$. Thus by acyclicity of $\mathcal{M}/\pi\mathcal{M}$ we get that the maps

$$\Gamma(\mathcal{M}/\pi^{n+1}\mathcal{M}) \to \Gamma(\mathcal{M}/\pi^n\mathcal{M})$$

are surjective for every $n \geq 1$, as required. □

Analogously to Lemma 4.3.5, we then have:

Proposition 5.3.9. *For any $w \in W$, the global section functor $\Gamma \circ (\widehat{f_w})_*$ is strict exact and objects of $\widehat{\mathcal{M}_{B_q}(G_q)}_w$ have Γ-acyclic image under $(\widehat{f_w})_*$.*

Proof. Suppose we have a strict short exact sequence

$$0 \to \mathcal{K} \to \mathcal{M} \to \mathcal{N} \to 0$$

in $\widehat{\mathcal{M}_{B_q}(G_q)}_w$. By Lemma 5.3.7(ii), we may assume that the unit ball $\mathcal{M}^\circ$ of $\mathcal{M}$ is in $\widehat{\mathscr{C}_R^w}$. By strictness, we may assume that $\mathcal{K}$ and $\mathcal{N} = \mathcal{M}/\mathcal{K}$ are equipped with the subspace and quotient topologies respectively. Note that from these assumptions, the comodule maps on $\mathcal{M}$, $\mathcal{K}$ and $\mathcal{N}$ all have norm at most 1, and so the unit balls are all in $\widehat{\mathscr{C}_R^w}$ by Lemma 5.1.10. Furthermore, we have a short exact sequence

$$0 \to \mathcal{K}^\circ \to \mathcal{M}^\circ \to \mathcal{N}^\circ \to 0$$

which induces short exact sequences

$$0 \to \mathcal{K}^\circ/\pi^n\mathcal{K}^\circ \to \mathcal{M}^\circ/\pi^n\mathcal{M}^\circ \to \mathcal{N}^\circ/\pi^n\mathcal{N}^\circ \to 0$$

for every $n \geq 1$ by strictness. Now $\mathcal{K}^\circ/\pi^n\mathcal{K}^\circ \in \mathscr{C}_R^w$ and so it is acyclic by Lemma 4.3.5. Hence we get a tower of short exact sequences

$$0 \to \Gamma(\mathcal{K}^\circ/\pi^n\mathcal{K}^\circ) \to \Gamma(\mathcal{M}^\circ/\pi^n\mathcal{M}^\circ) \to \Gamma(\mathcal{N}^\circ/\pi^n\mathcal{N}^\circ) \to 0.$$

Moreover, since $\mathcal{K}^\circ$ has no π-torsion, we may apply Lemma 5.3.8 to obtain that $(\Gamma(\mathcal{K}^\circ/\pi^n\mathcal{K}^\circ))_{n\geq 1}$ satisfies the Mittag-Leffler condition. Thus, by applying Proposition 5.2.4, we can pass to the inverse limit to get a short exact sequence

$$0 \to \Gamma(\mathcal{K}^\circ) \to \Gamma(\mathcal{M}^\circ) \to \Gamma(\mathcal{N}^\circ) \to 0.$$

Extending scalars to L and applying Proposition 5.3.4(i), we get that

$$0 \to \Gamma(\mathcal{K}) \to \Gamma(\mathcal{M}) \to \Gamma(\mathcal{N}) \to 0.$$

is algebraically exact, hence strict exact by Example 2.9.6.

The last part follows exactly as in Lemma 4.3.5, using the fact that $\widehat{f_w}^*$ is strict exact by Corollary 5.3.6(ii) and so its right adjoint preserves injectives by Lemma 2.9.9. □

We are now in a position to construct a Čech complex and prove that it computes the cohomology. Just as in the discussion after Proposition 5.2.9, we may define an augmented complex C^{aug} and an augmented standard complex $\check{C}^{\text{aug}}$ as well as their non-augmented counter-parts C and $\check{C}$.

Theorem 5.3.10. *For any $\mathcal{M} \in \widehat{\mathcal{M}_{B_q}(G_q)}$, the complex $C^{aug}(\mathcal{M})$ is strict exact and is an acyclic resolution of $\mathcal{M}$ in $\widehat{\mathcal{M}_{B_q}(G_q)}$.*

Proof. The fact that the terms in $C(\mathcal{M})$ are Γ-acyclic follows from Proposition 5.3.9. So we are left to show that $C^{\text{aug}}(\mathcal{M})$ is strict exact. By Proposition 5.3.4 we may pick $F_0\mathcal{M} \in \widehat{\mathscr{C}_R}$ such that $\mathcal{M} = (F_0\mathcal{M})_L$. Also, for any torsion-free $\mathcal{N} \in \widehat{\mathscr{C}_R}$ we have that $(\widehat{f_w})^*(\mathcal{N}_L) \cong (\widehat{f_w})^*(\mathcal{N}) \otimes_R L$ by Corollary 5.3.6(i). Thus we see that

$$C^{\text{aug}}(\mathcal{M}) = C^{\text{aug}}(F_0\mathcal{M}) \otimes_R L$$

and it will suffice to show that $C^{\text{aug}}(F_0\mathcal{M})$ is exact (algebraically).

Now we have that

$$C^{\text{aug}}(F_0\mathcal{M}) \cong \varprojlim C^{\text{aug}}(F_0\mathcal{M}/\pi^n F_0\mathcal{M})$$

by Remark 5.2.10. However the complexes $C^{\text{aug}}(F_0\mathcal{M}/\pi^n F_0\mathcal{M})$ are exact for all $n \geq 1$ by Lemma 5.2.2 and Proposition 4.3.7. Moreover the maps

$$C^{\text{aug}}(F_0\mathcal{M}/\pi^{n+1} F_0\mathcal{M}) \to C^{\text{aug}}(F_0\mathcal{M}/\pi^n F_0\mathcal{M})$$

are all surjective, so the projective system of complexes satisfies the Mittag-Leffler condition. The induced maps between the cohomology groups of thoses complexes, which are all zero, trivially also satisfy the Mittag-Leffler condition. Hence we may apply Proposition 5.2.12 to get that the cohomology groups of $C^{\mathrm{aug}}(F_0\mathcal{M})$ are all zero as required. $\square$

From this we may deduce our promised Theorem D that the standard complex computes the cohomology of Γ:

Proof of Theorem D. This now follows immediately from the previous Theorem by Proposition 2.9.13(iii). $\square$

Corollary 5.3.11. *The global section functor on $\widehat{\mathcal{M}_{B_q}(G_q)}$ has finite cohomological dimension. More specifically, $R^i\Gamma = 0$ for $i > N+1$.*

Proof. Take $\mathcal{M} \in \widehat{\mathcal{M}_{B_q}(G_q)}$. By Theorem D, the cohomology $R\Gamma(\mathcal{M})$ is computed by the complex $\check{C}(\mathcal{M})$. Now let $F_0\mathcal{M} \in \widehat{\mathscr{C}_R}$ be the lattice in $\mathcal{M}$ given by Proposition 5.3.4, so that $\check{C}(\mathcal{M}) = \check{C}(F_0\mathcal{M}) \otimes_R L$, and we have

$$\check{C}(F_0\mathcal{M}) \cong \varprojlim \check{C}(F_0\mathcal{M}/\pi^n F_0\mathcal{M}).$$

Now $C(F_0\mathcal{M})$ is a complex of the form

$$C(F_0\mathcal{M}) : \mathcal{N}_1 \to \mathcal{N}_2 \to \cdots$$

such that each $\mathcal{N}_i$ is π-torsion free by Corollary 2.8.15 and, for every $n \geq 1$, each $\mathcal{N}_i/\pi^n\mathcal{N}_i$ is Γ-acyclic by Lemma 4.3.5. Hence the projective system given by the complexes

$$\check{C}(F_0\mathcal{M}/\pi^n F_0\mathcal{M}) : \Gamma(\mathcal{N}_1/\pi^n\mathcal{N}_1) \to \Gamma(\mathcal{N}_2/\pi^n\mathcal{N}_2) \to \cdots$$

satisfies the Mittag-Leffler condition by Lemma 5.3.8. Moreover, for every $n \geq 1$, the i-th cohomology group of $\check{C}(F_0\mathcal{M}/\pi^n F_0\mathcal{M})$ is zero for $i > N$ by Proposition 4.3.7 and Proposition 4.2.8(iv). Thus the induced maps

$$R^i\Gamma(F_0\mathcal{M}/\pi^{n+1}F_0\mathcal{M}) \to R^i\Gamma(F_0\mathcal{M}/\pi^n F_0\mathcal{M})$$

trivially satisfy the Mittag-Leffler condition for $i > N$, and we may therefore invoke Proposition 5.2.12 to obtain that the complex $\check{C}(F_0\mathcal{M})$ is exact in degrees bigger than $N+1$. Hence the complex $\check{C}(\mathcal{M})$ is algebraically exact, i.e. strict exact, in degrees bigger than $N+1$ and the result now follows by Lemma 2.9.12(iii). $\square$

5.4 The Beilinson-Bernstein Theorem for $\widehat{\mathcal{D}^{\lambda}_{B_q}(G_q)}$

In this Section, we define suitable notions of D-modules on our analytic quantum flag varieties and apply the results from the previous Sections to obtain a Beilinson-Bernstein localisation theorem.

Recall the notion of a module algebra over a Hopf algebra from Definition 2.1.15. If H denotes an R-Hopf algebra, then its π-adic completion $\widehat{H}$ may be thought of as a Hopf algebra-like object, with maps $\widehat{\Delta} : \widehat{H} \to H\widehat{\otimes}_R H$, $\widehat{\varepsilon} : \widehat{H} \to R$ and $\widehat{S} : \widehat{H} \to \widehat{H}$ satisfying the usual axioms. We will then say that an R-algebra A which is also an $\widehat{H}$-module is an $\widehat{H}$-module algebra if, when viewed as an H-module, it is an H-module algebra.

We now define those elements of $\widehat{\mathscr{C}_R}$ which play the role of D-modules. Recall that, by Lemma 5.1.2, if $\mathcal{M}$ is a $\widehat{\mathcal{B}_q}$-comodule then it is a $\widehat{U^{\text{res}}(\mathfrak{b})}$-module. Also note that the inclusion $U^{\geq 0} \subseteq U^{\text{res}}(\mathfrak{b})$ induces an R-algebra homomorphism $\widehat{U^{\geq 0}} \to \widehat{U^{\text{res}}(\mathfrak{b})}$ so that $\widehat{U^{\text{res}}(\mathfrak{b})}$-modules are naturally $\widehat{U^{\geq 0}}$-modules.

Note that $\mathcal{A}_q/\pi^a\mathcal{A}_q$ is a U/π^aU-module algebra for all $a \geq 1$. Hence we see that $\widehat{\mathcal{A}_q}$ is a $\widehat{U}$-module algebra. Thus, since $\widehat{\mathcal{D}} = \widehat{\mathcal{A}_q}\widehat{\otimes}_R\widehat{U}$ as an R-module, we see that we may think of $\widehat{\mathcal{D}}$ as the smash product algebra of $\widehat{\mathcal{A}_q}$ and $\widehat{U}$. Moreover it is a $\widehat{U}$-module algebra. Similarly to the discussion preceeding Definition 4.4.2, we then see that $\widehat{\mathcal{D}}$ is a $\widehat{U^{\text{res}}}$-module algebra since $\mathcal{D}$ is a U^{res}-module algebra. Finally observe that the inclusion $U^{\geq 0} \to \mathcal{D}$ induces a map $\widehat{U^{\geq 0}} \to \widehat{\mathcal{D}}$, so that every $\widehat{\mathcal{D}}$-module is naturally a $\widehat{U^{\geq 0}}$-module.

Definition 5.4.1. Let $\lambda \in T_P^R$. We define the category $\widehat{\mathscr{D}^\lambda}$ to have objects triples $(\mathcal{M}, \alpha, \beta)$ where $\mathcal{M}$ is a π-adically complete R-module, $\alpha : \widehat{\mathcal{D}}\widehat{\otimes}_R\mathcal{M} \to \mathcal{M}$ is a $\widehat{\mathcal{D}}$-module structure, and $\beta : \mathcal{M} \to \mathcal{M}\widehat{\otimes}_R\mathcal{B}_q$ is a $\widehat{\mathcal{B}_q}$-comodule structure. By the above this induces a $\widehat{U^{\text{res}}(\mathfrak{b})}$-module structure on $\mathcal{M}$, which we also denote by β. We require these to satisfy the following:

(i) the $\widehat{U^{\geq 0}}$-actions on $\mathcal{M} \otimes_R R_\lambda$ given by $\beta \otimes \lambda$ and $\alpha|_{U^{\geq 0}} \otimes 1$ are equal; and

(ii) the action map $\alpha : \mathcal{D}\widehat{\otimes}_R\mathcal{M} \to \mathcal{M}$ is $\widehat{U^{\text{res}}(\mathfrak{b})}$-linear.

The morphisms are the R-module maps which are both $\widehat{\mathcal{B}_q}$-comodule homomorphisms and $\widehat{\mathcal{D}}$-module homomorphisms. As usual we call $\mathcal{M} \in \widehat{\mathscr{D}^\lambda}$ *coherent* if it is finitely generated as $\widehat{\mathcal{D}}$-module, and denote by $\text{coh}(\widehat{\mathscr{D}^\lambda})$ the full subcategory of such objects.

As always there is a forgetful functor $\widehat{\mathscr{D}^\lambda} \to \widehat{\mathscr{C}_R}$ and we define the global sections of $\mathcal{M} \in \widehat{\mathscr{D}^\lambda}$ to be its global sections as an object of $\widehat{\mathscr{C}_R}$. Moreover, analogously to Lemma 5.2.2, if $\mathcal{M} \in \widehat{\mathscr{D}^\lambda}$ then $\mathcal{M}/\pi^n\mathcal{M} \in \mathscr{D}^\lambda$ for every $n \geq 1$.

We saw earlier that the π-adic completion functor sends the category $\mathscr{C}_R$ to the category $\widehat{\mathscr{C}_R}$, and it is straightforward to see that π-adic completion will also send the category $\mathscr{D}^\lambda$ to $\widehat{\mathscr{D}^\lambda}$. In particular we have a "structure sheaf" $\widehat{\mathcal{D}^\lambda}$. We now check that it represents global sections, which in particular gives that $\Gamma(\widehat{\mathcal{D}^\lambda})$ is a ring.

Lemma 5.4.2. *Let $\mathcal{M} \in \widehat{\mathscr{D}^\lambda}$. Then $\Gamma(\mathcal{M}) \cong \text{Hom}_{\widehat{\mathscr{D}^\lambda}}(\widehat{\mathcal{D}^\lambda}, \mathcal{M})$.*

Proof. By Proposition 5.2.4 we have isomorphisms

$$\begin{aligned}\Gamma(\mathcal{M}) \cong \varprojlim \Gamma(\mathcal{M}/\pi^a\mathcal{M}) &\cong \varprojlim \text{Hom}_{\mathscr{D}^\lambda}(\mathcal{D}^\lambda, \mathcal{M}/\pi^a\mathcal{M}) \\ &\cong \varprojlim \text{Hom}_{\mathscr{D}^\lambda}(\mathcal{D}^\lambda/\pi^a\mathcal{D}^\lambda, \mathcal{M}/\pi^a\mathcal{M})\end{aligned}$$

since global sections in $\mathscr{D}^\lambda$ are represented by $\mathcal{D}^\lambda$. Now the same argument as in the proof of Proposition 5.2.4 shows that the latter is isomorphic to $\text{Hom}_{\widehat{\mathscr{D}^\lambda}}(\widehat{\mathcal{D}^\lambda}, \mathcal{M})$. □

Recall that the functor $M \mapsto M(\mu)$ sends objects in $\mathscr{D}^\lambda$ to $\mathscr{D}^{\lambda+\mu}$. Given $\mathcal{M} \in \widehat{\mathscr{C}_R}$ and $\mu \in T_P^R$, we define $\mathcal{M}(\mu) := \mathcal{M} \otimes_R R_{-\mu}$ which belongs to $\widehat{\mathscr{C}_R}$ with the tensor comodule structure (since $R_{-\mu}$ is π-adically complete, it is a $\widehat{\mathcal{B}_q}$-comodule).

Notation. Given $\mathcal{M} \in \widehat{\mathscr{C}_R}$, we write $\operatorname{gr}_0 \mathcal{M} := \mathcal{M}/\pi\mathcal{M} \in \mathscr{C}_R$ (by Lemma 5.2.2). Following [6, Section 2.7] we call $\operatorname{gr}_0 \mathcal{M}$ the *slice* of $\mathcal{M}$.

Before we state our next result, we need a preparatory elementary fact:

Lemma 5.4.3. *If D is a π-adically complete Noetherian R-algebra and M is a finitely generated D-module, then M is π-adically complete. Moreover if $\pi M = M$, then $M = 0$.*

Proof. By [17, 3.2.3], $\widehat{M} \cong \widehat{D} \otimes_D M \cong D \otimes_D M \cong M$ so that M is π-adically complete. If now $M = \pi M$ then $M = \pi^n M$ for all $n \geq 1$ and so $M \cong \widehat{M} = \varprojlim M/\pi^n M = 0$. □

Theorem 5.4.4. *Suppose $\mathcal{M} \in \operatorname{coh}(\widehat{\mathscr{D}^{\lambda}})$ is π-torsion free. Then there is a surjection*

$$\left(\widehat{\mathcal{D}^{\lambda+\mu}}(-\mu)\right)^a \twoheadrightarrow \mathcal{M}$$

for some $a \geq 1$ and some $\mu \in P$.

Proof. By Theorem 4.4.6, $(\operatorname{gr}_0 \mathcal{M})(\mu)$ is Γ-acyclic and generated by finitely many global sections for $\mu >> 0$. Fix such a μ. Then there is a map $(\mathcal{D}^{\lambda+\mu})^a \twoheadrightarrow (\operatorname{gr}_0 \mathcal{M})(\mu)$ in $\mathscr{D}^{\lambda+\mu}$ for some $a \geq 1$. Now let $\mathcal{N} = \mathcal{M}(\mu)$. Since $\mathcal{M}$ has no π-torsion, so does $\mathcal{N}$ and we obtain a surjection

$$\Gamma(\mathcal{N}/\pi^{n+1}\mathcal{N}) \twoheadrightarrow \Gamma(\mathcal{N}/\pi^n\mathcal{N}) \quad \forall n \geq 1$$

by Lemma 5.3.8 since $(\operatorname{gr}_0 \mathcal{M})(\mu)$ is Γ-acyclic. Hence, starting from a generating set of $(\operatorname{gr}_0 \mathcal{M})(\mu)$ in its global sections, we can inductively construct $w_1, \ldots, w_a \in \varprojlim \Gamma(\mathcal{N}/\pi^n\mathcal{N})$. Since $\Gamma(\mathcal{N}) \cong \varprojlim \Gamma(\mathcal{N}/\pi^n\mathcal{N})$ by Proposition 5.2.4, these elements correspond to global sections in $\mathcal{N}$ and they define a map $(\widehat{\mathcal{D}^{\lambda+\mu}})^a \to \mathcal{N}$ which must be surjective by the previous Lemma, since it is surjective modulo π. Thus, twisting by $-\mu$, we obtain a map

$$\left(\widehat{\mathcal{D}^{\lambda+\mu}}(-\mu)\right)^a \twoheadrightarrow \mathcal{M}$$

as required. □

We now move on to define a suitable category of D-modules in $\widehat{\mathcal{M}_{B_q}(G_q)}$. We let $\widehat{\mathcal{D}_q} := \widehat{\mathcal{D}} \otimes_R L$ and $\widehat{U_q^{\geq 0}} := \widehat{U^{\geq 0}} \otimes_R L$. Suppose that A is an L-Banach algebra, and H is a torsion-free R-Hopf algebra. We will say that A is a $\widehat{H_L}$-module algebra if, viewed as a H_L-module, it is a module algebra. In such a situation we may then define the smash product algebra $\widehat{H_L \# A}$ to be the completion of the smash product $H_L \# A$.

After extending scalars, we see from our discussion preceeding Definition 5.4.1 that $\widehat{\mathcal{O}_q}$ is a $\widehat{U_q}$-module algebra, $\widehat{\mathcal{D}_q} \cong \widehat{\mathcal{O}_q \# U_q}$ is a $\widehat{U_L^{\text{res}}}$-module algebra and there is a continuous algebra homomorphism $\widehat{U_q^{\geq 0}} \to \widehat{\mathcal{D}_q}$. Moreover we have a contractive (i.e. bounded of norm at most 1) algebra homomorphism $\widehat{U_q^{\geq 0}} \to \widehat{U^{\text{res}}(\mathfrak{b})_L}$, and so any Banach $\widehat{U^{\text{res}}(\mathfrak{b})_L}$-module will also be a Banach $\widehat{U_q^{\geq 0}}$-module. We've already observed that R_λ is a $\widehat{\mathcal{B}_q}$-comodule. Extending scalars, that makes L_λ into a $\widehat{\mathcal{O}_q(B)}$-comodule. Finally, recall from Theorem C that an $\widehat{\mathcal{O}_q(B)}$-comodule is the same thing as a topologically integrable $\widehat{U^{\text{res}}(\mathfrak{b})_L}$-module, and comodule homomorphisms are just $\widehat{U^{\text{res}}(\mathfrak{b})_L}$-linear maps.

Definition 5.4.5. Let $\lambda \in T_P^R$. A (B_q, λ)-*equivariant* $\widehat{\mathcal{D}_q}$-*module* is a triple $(\mathcal{M}, \alpha, \beta)$ where $\mathcal{M}$ is an L-Banach space, $\alpha : \widehat{\mathcal{D}_q}\widehat{\otimes}_L\mathcal{M} \to \mathcal{M}$ is a left $\widehat{\mathcal{D}_q}$-module action and $\beta : \mathcal{M} \to \mathcal{M}\widehat{\otimes}_L\widehat{\mathcal{O}_q(B)}$ is a right $\widehat{\mathcal{O}_q(B)}$-comodule action. The map β induces a left $\widehat{U^{\mathrm{res}}(\mathfrak{b})_L}$-action on $\mathcal{M}$ which we also denote by β. These actions must satisfy:

(i) The $\widehat{U_q^{\geq 0}}$-actions on $\mathcal{M} \otimes_L L_\lambda$ given by $\beta \otimes \lambda$ and $\alpha|_{\widehat{U_q^{\geq 0}}} \otimes 1$ are equal.

(ii) The map α is $\widehat{U^{\mathrm{res}}(\mathfrak{b})_L}$-linear with respect to the β-action on $\mathcal{M}$ and the above action on $\widehat{\mathcal{D}_q}$.

The above form a category which we denote by $\widehat{\mathcal{D}^\lambda_{B_q}(G_q)}$. We call an object $\mathcal{M} \in \widehat{\mathcal{D}^\lambda_{B_q}(G_q)}$ *coherent* if it is finitely generated as a $\widehat{\mathcal{D}_q}$-module. We denote by $\mathrm{coh}(\widehat{\mathcal{D}^\lambda_{B_q}(G_q)})$ the full subcategory of $\widehat{\mathcal{D}^\lambda_{B_q}(G_q)}$ consisting of coherent modules. Since $\widehat{\mathcal{D}_q}$ is Noetherian by Proposition 4.4.1, this category is abelian.

Note that condition (ii) in the definition above implies that the restriction of α to $\widehat{\mathcal{O}_q}\widehat{\otimes}_L\mathcal{M}$ is $U^{\mathrm{res}}(\mathfrak{b})_L$-linear. Since both sides are topologically integrable, this map is therefore a comodule homomorphism. Thus we see that there is a forgetful functor $\widehat{\mathcal{D}^\lambda_{B_q}(G_q)} \to \widehat{\mathcal{M}_{B_q}(G_q)}$.

Recall from Lemma 2.8.6 that any object $\mathcal{M} \in \mathrm{coh}(\widehat{\mathcal{D}^\lambda_{B_q}(G_q)})$ is canonically equipped with the quotient topology coming from a surjection $(\widehat{\mathcal{D}_q})^a \to \mathcal{M}$ given by a finite set of generators.

Theorem 5.4.6. *Let $\mathcal{M} \in \widehat{\mathcal{D}^\lambda_{B_q}(G_q)}$. Then $\mathcal{M}$ contains an R-lattice $F_0\mathcal{M}$ which is an element of $\widehat{\mathscr{D}^\lambda}$. Moreover, $F_0\mathcal{M}$ can be chosen to be an element of* $\mathrm{coh}(\widehat{\mathscr{D}^\lambda})$ *if* $\mathcal{M} \in \mathrm{coh}(\widehat{\mathcal{D}^\lambda_{B_q}(G_q)})$.

Proof. By Proposition 5.3.4(ii) and its proof, there exists some R-lattice $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{N} \in \widehat{\mathscr{C}_R}$ and there is some $m \geq 0$ such that

$$\mathcal{N} \subseteq \pi^{-m}\mathcal{M}^\circ. \tag{5.1}$$

We now define $F_0\mathcal{M}$ to be the closure of $\widehat{\mathcal{D}} \cdot \mathcal{N}$. By construction it is a $\widehat{\mathcal{D}}$-module and contains $\mathcal{N}$. Moreover, since $\mathcal{M}^\circ$ is a closed $\widehat{\mathcal{D}}$-submodule of $\mathcal{M}$, we get from (5.1) that

$$F_0\mathcal{M} \subseteq \pi^{-m}\mathcal{M}^\circ. \tag{5.2}$$

So we see that $F_0\mathcal{M}$ is a closed lattice inside $\mathcal{M}$, hence is π-adically complete. We just need to show that it is also a $\widehat{\mathcal{B}_q}$-comodule, as the compatibility condition will be inherited from the one in $\mathcal{M}$. Note that the $\widehat{\mathcal{D}}$-action on $\mathcal{M}$ is $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-linear by the axioms of $\widehat{\mathcal{D}^\lambda_{B_q}(G_q)}$, and so we see that $F_0\mathcal{M}$ is a $\widehat{U^{\mathrm{res}}(\mathfrak{b})}$-submodule of $\mathcal{M}$.

Let $\mathcal{K} := \mathcal{D} \cdot \mathcal{N}$, which is a $U^{\mathrm{res}}(\mathfrak{b})$-module. Since the action of any E_{α_i} on $\mathcal{M}$ is locally topologically nilpotent, we see that for any $a \geq 1$, the action of any $E_{\alpha_i}^{(r)}$ on any element of $F_0\mathcal{M}/\pi^a F_0\mathcal{M}$ is zero for $r >> 0$. Note that every element of $F_0\mathcal{M}/\pi^a F_0\mathcal{M}$ can be represented by an element of $\mathcal{K}$ since $\mathcal{K}$ is dense in $F_0\mathcal{M}$. The PBW basis in

$U^{\pm}$ (see Corollary 3.3.7) consists of weight vectors for the adjoint action of $(U^{\text{res}})^0$, and every element of $\mathcal{A}_q$ is a finite sum of weight vectors since $\mathcal{A}_q$ is integrable. Thus, by the triangular decomposition for U (Remark 3.3.8), we see that every element of $\mathcal{D}$ is a finite sum of weight vectors for the action of U^{res}. Moreover, by Proposition 5.1.8, $\mathcal{N}$ is topologically semisimple and so every element is a convergent sum of weight vectors, hence a finite sum of weight vectors modulo π^a. Thus every element of $\mathcal{K}$ has image in $F_0\mathcal{M}/\pi^a F_0\mathcal{M}$ which is a finite sum of weight vectors. Thus we see that $F_0\mathcal{M}/\pi^a F_0\mathcal{M}$ is integrable, i.e. a $\mathcal{B}_q$-comodule. Passing to the inverse limit, we see that $F_0\mathcal{M}$ is a $\widehat{\mathcal{B}_q}$-comodule as required.

Now if $\mathcal{M}$ is coherent, then $\mathcal{M}^\circ$ is a quotient of a finite direct sum of $\widehat{\mathcal{D}}$'s. Since $\mathcal{M}^\circ$ is a finitely generated module, we see from (5.2) that $F_0\mathcal{M}$ is as well as $\widehat{\mathcal{D}}$ is Noetherian by Proposition 4.4.1 (in fact $F_0\mathcal{M} = \widehat{\mathcal{D}} \cdot \mathcal{N}$ in this case since the right hand side is then finitely generated and so π-adically complete by [17, 3.2.3(v)]). $\square$

The functor $- \otimes_R L$ maps $\operatorname{coh}(\widehat{\mathscr{D}^\lambda})$ to $\operatorname{coh}(\widehat{\mathcal{D}^\lambda_{B_q}(G_q)})$ and so by pre-composing with the functor of taking π-adic completion, we obtain a functor $M \mapsto \widehat{M_L}$ from $\operatorname{coh}(\mathscr{D}^\lambda)$ to $\operatorname{coh}(\widehat{\mathcal{D}^\lambda_{B_q}(G_q)})$. Hence in particular we have an object $\widehat{\mathcal{D}^\lambda_q} := \widehat{\mathcal{D}^\lambda_L}$.

Lemma 5.4.7. *The global section functor on $\widehat{\mathcal{D}^\lambda_{B_q}(G_q)}$ is represented by $\widehat{\mathcal{D}^\lambda_q}$.*

Proof. By Theorem 5.4.6, given a module $\mathcal{M} \in \widehat{\mathcal{D}^\lambda_{B_q}(G_q)}$, we have a lattice $F_0\mathcal{M} \subseteq \mathcal{M}$ such that $F_0\mathcal{M}$ is in $\widehat{\mathscr{D}^\lambda}$. The result therefore follows by Proposition 5.3.4(i) and Lemma 5.4.2. $\square$

The next few results are analogous to [6, Theorem 6.6].

Theorem 5.4.8. *Suppose that $M \in \operatorname{coh}(\mathscr{D}^\lambda)$ is such that M_L is generated by its global sections as a $\mathcal{D}_q$-module. Then $\widehat{M_L}$ is generated by its global sections.*

Proof. Since $\mathcal{D}_q$ is Noetherian by Proposition 4.1.6 and since M_L is finitely generated as a $\mathcal{D}_q$-module, we can find $m_1, \dots, m_a \in \Gamma(M_L)$ generating M_L as a $\mathcal{D}_q$-module. By rescaling we may assume that $m_1, \dots, m_a$ lie in $\Gamma(M)$ as $\Gamma(M_L) = \Gamma(M) \otimes_R L$ by Proposition 4.3.10. Let $\alpha : (\mathcal{D}^\lambda)^a \to M$ be the map in $\operatorname{coh}(\mathscr{D}^\lambda)$ defined by these global sections, let $N = \operatorname{Im}\alpha$, and let $C = \operatorname{coker}(\alpha) \in \operatorname{coh}(\mathscr{D}^\lambda)$. By assumption, $C_L = 0$ and thus there is some $m \geq 0$ such that $\pi^m C = 0$. In other words, $\pi^m M \subseteq N$.

Now we have a short exact sequence

$$0 \to N \to M \to C \to 0$$

in $\operatorname{coh}(\mathscr{D}^\lambda)$, which induces a tower of short exact sequences

$$0 \to (N + \pi^n M)/\pi^n M \to M/\pi^n M \to C/\pi^n C \to 0$$

for $n \geq 1$. Since the inverse system $((N+\pi^n M)/\pi^n M)_n$ trivially satisfies the Mittag-Leffler condition, passing to the inverse limit yields a short exact sequence

$$0 \to \varprojlim (N + \pi^n M)/\pi^n M \to \widehat{M} \to \widehat{C} \to 0.$$

On the other hand, since $\pi^m M \subseteq N$, it follows that for every $n \geq m$, we have $\pi^n N \subseteq N \cap \pi^n M \subseteq \pi^{n-m} N$. Thus we obtain that $\widehat{N} \cong \varprojlim (N + \pi^n M)/\pi^n M$. Moreover, since taking π-adic completion preserves surjections, $(\widehat{\mathcal{D}^\lambda})^a$ surjects onto $\widehat{N}$. Putting everything together, this gives that the sequence

$$(\widehat{\mathcal{D}^\lambda})^a \xrightarrow{\widehat{\alpha}} \widehat{M} \to \widehat{C} \to 0$$

is also exact. But since $\pi^m C = 0$, we have that $\pi^m \widehat{C} = 0$ also, and thus $\widehat{C} \otimes_R L = 0$. Therefore by applying the exact functor $- \otimes_R L$ to the above exact sequence, we deduce that $\widehat{\alpha}_L : (\widehat{\mathcal{D}_q^\lambda})^a \to \widehat{M_L}$ is surjective, and the result follows by Lemma 5.4.7. □

Corollary 5.4.9. *Suppose that $\lambda \in T_P^R$ is regular and dominant. If $\mathcal{M} \in \mathrm{coh}(\widehat{\mathcal{D}_{B_q}^\lambda(G_q)})$ then $\mathcal{M}$ is generated by its global sections.*

Proof. By Theorem 5.4.6 there is a lattice $F_0\mathcal{M}$ in $\mathcal{M}$ which is an element of $\mathrm{coh}(\mathscr{D}^\lambda)$. By Theorem 5.4.4 there is a surjection

$$\left(\widehat{\mathcal{D}^{\lambda+\mu}(-\mu)}\right)^a \twoheadrightarrow F_0\mathcal{M}$$

By extending scalars we get a surjection

$$\left(\widehat{\mathcal{D}_q^{\lambda+\mu}(-\mu)}\right)^a \twoheadrightarrow \mathcal{M}.$$

But by Theorem 5.4.8, which applies by the proof of [13, Theorem 4.12] (which shows that for λ regular dominant, every object of $\mathcal{D}_{B_q}^\lambda(G_q)$ is generated by its global sections), $\widehat{\mathcal{D}_q^{\lambda+\mu}(-\mu)}$ is generated by its global sections. Hence so is $\mathcal{M}$ by Lemma 5.4.7. □

We can now prove that global sections is exact on $\mathrm{coh}(\widehat{\mathcal{D}_{B_q}^\lambda(G_q)})$.

Theorem 5.4.10. *For any $\lambda \in T_P^R$, $\widehat{\mathcal{D}_q^\lambda}$ is Γ-acyclic.*

Proof. By Proposition 4.4.4 and Lemma 5.3.8, the projective system $(R^i\Gamma(\mathcal{D}^\lambda/\pi^n\mathcal{D}^\lambda))_n$ satisfies the Mittag-Leffler condition for each $i \geq 0$. So by Proposition 5.2.12 applied to $(\check{C}^{\mathrm{aug}}(\mathcal{D}^\lambda/\pi^n\mathcal{D}^\lambda))_n$ and by Proposition 4.3.7 combined with Remark 5.2.10, we get that

$$\check{H}^i(\widehat{\mathcal{D}^\lambda}) = H^i\check{C}^{\mathrm{aug}}(\widehat{\mathcal{D}^\lambda}) \cong \varprojlim R^i\Gamma(\mathcal{D}^\lambda/\pi^n\mathcal{D}^\lambda)$$

for all $i \geq 1$. Hence by Proposition 4.4.4, $\check{H}^i(\widehat{\mathcal{D}^\lambda}) = 0$, and so by Theorem D we have that

$$R^i\Gamma(\widehat{\mathcal{D}_q^\lambda}) \cong H^i\check{C}^{\mathrm{aug}}(\widehat{\mathcal{D}^\lambda}) \otimes_R L = 0$$

for $i \geq 1$ as required. □

Corollary 5.4.11. *Suppose λ is regular and dominant. If $\mathcal{M} \in \mathrm{coh}(\widehat{\mathcal{D}_{B_q}^\lambda(G_q)})$ then $\mathcal{M}$ is Γ-acyclic and $\Gamma(\mathcal{M})$ is a finitely presented module over $D := \Gamma(\widehat{\mathcal{D}_q^\lambda})$.*

Proof. By Corollary 5.4.9 and since $\widehat{\mathcal{D}_q}$ is Noetherian by Proposition 4.4.1, we can find a resolution

$$\mathcal{F}_N \xrightarrow{f_N} \mathcal{F}_{N-1} \xrightarrow{f_{N-1}} \cdots \xrightarrow{f_1} \mathcal{F}_0 \xrightarrow{f_0} \mathcal{M} \to 0$$

where the $\mathcal{F}_i$ are all of the form $(\widehat{\mathcal{D}^\lambda_q})^a$. Then by Theorem 5.4.10 each $\mathcal{F}_i$ is Γ-acyclic. Let $\mathcal{M}_i = \ker f_i$. Then by the long exact sequence on cohomology, we get

$$R^j\Gamma(\mathcal{M}) = R^{j+1}\Gamma(\mathcal{M}_0) = R^{j+2}\Gamma(\mathcal{M}_1) = \cdots = R^{j+N+1}\Gamma(\mathcal{M}_N) = 0$$

for every $j > 0$ by Corollary 5.3.11. So $\mathcal{M}$ is Γ-acyclic.

For the last part, as above we have a resolution

$$(\widehat{\mathcal{D}^\lambda_q})^a \to (\widehat{\mathcal{D}^\lambda_q})^b \to \mathcal{M} \to 0$$

for some $a, b \geq 0$. By the above the global section functor is exact on $\mathrm{coh}(\widehat{\mathcal{D}^\lambda_{B_q}(G_q)})$, so it follows that we have a resolution

$$D^a \to D^b \to \Gamma(\mathcal{M}) \to 0$$

making $\Gamma(\mathcal{M})$ finitely presented as required. □

By Corollary 5.4.11, we get a functor

$$\Gamma : \mathrm{coh}(\widehat{\mathcal{D}^\lambda_{B_q}(G_q)}) \to \text{f.p } D\text{-mod}.$$

Now let M be a finitely presented (or even finitely generated) D-module. Then we define its *localisation* to be the $\widehat{\mathcal{D}_q}$-module

$$\mathrm{Loc}_\lambda(M) = \widehat{\mathcal{D}^\lambda_q} \otimes_D M.$$

Note that this is a finitely generated $\widehat{\mathcal{D}_q}$-module so is automatically Banach, and in fact a quotient of a finite direct sum of $\widehat{\mathcal{D}^\lambda_q}$. This defines a well-defined comodule structure on $\mathrm{Loc}_\lambda(M)$ because D is a trivial subcomodule of $\widehat{\mathcal{D}^\lambda_q}$. This therefore defines a functor

$$\mathrm{Loc}_\lambda : \text{f.p } D\text{-mod} \to \mathrm{coh}(\widehat{\mathcal{D}^\lambda_{B_q}(G_q)}),$$

which is a left adjoint to global section. Indeed the adjunction morphisms are given as follows: the map $\mathrm{Loc}_\lambda(\Gamma(\mathcal{M})) \to \mathcal{M}$ is given by the $\widehat{\mathcal{D}_q}$-action and is well-defined because $\widehat{\mathcal{D}^\lambda_q}$ represents global sections, while the map $M \to \Gamma(\mathrm{Loc}_\lambda(M))$ is just the natural map $m \mapsto 1 \otimes m$.

We can now finally prove our localisation theorem:

Proof of Theorem E. The proof is now standard. We have to show that the two adjunction morphisms are isomorphisms. We first show that the adjunction morphism

$$\psi_M : M \to \Gamma(\mathrm{Loc}_\lambda(M))$$

is an isomorphism for any finitely presented D-module M. By definition of Loc_λ, it is clear that ψ_M is an isomorphism whenever M is a free D-module. For a general finitely presented M, we have a presentation $D^a \to D^b \to M \to 0$. Since the global sections functor is exact by Corollary 5.4.11 and since Loc_λ is right exact, applying $\Gamma \circ \mathrm{Loc}_\lambda$ to this presentation and the Five Lemma yields that ψ_M is an isomorphism for any finitely presented module M.

It remains to show that the other adjunction morphism

$$\phi_{\mathcal{M}} : \mathrm{Loc}_\lambda(\Gamma(\mathcal{M})) \to \mathcal{M}$$

is an isomorphism for every $\mathcal{M} \in \mathrm{coh}(\widehat{\mathcal{D}^\lambda_{B_q}(G_q)})$. By exactness of Γ and since every object of $\mathrm{coh}(\widehat{\mathcal{D}^\lambda_{B_q}(G_q)})$ is generated by its global sections by Corollary 5.4.9, the map $\phi_{\mathcal{M}}$ is an isomorphism if and only if $\Gamma(\phi_{\mathcal{M}})$ is an isomorphism. But the composite

$$\Gamma(\mathcal{M}) \xrightarrow{\psi_{\Gamma(\mathcal{M})}} \Gamma(\mathrm{Loc}_\lambda(\Gamma(\mathcal{M}))) \xrightarrow{\Gamma(\phi_{\mathcal{M}})} \Gamma(\mathcal{M})$$

is the identity map by general properties of adjunctions (see [62, Theorem IV.1.1]), and the map $\psi_{\Gamma(\mathcal{M})}$ is an isomorphism by the above, therefore $\Gamma(\phi_{\mathcal{M}})$ is an isomorphism. □

Recall that a ring is called left coherent if every finitely generated left ideal is finitely presented.

Corollary 5.4.12. *The ring D is left coherent.*

Proof. By Theorem E, the category of finitely presented left D-modules is abelian. Thus it is closed under taking kernels. So, if I is a finitely generated left ideal in D, then the surjection $D \to D/I$ is a map between finitely presented modules whose kernel must be finitely presented as well. Hence I is finitely presented. □

From now on, we assume that the result [13, Proposition 4.8] holds.

From what we have done, the only thing left to do is to identify the ring D of global sections. Recall the definition of the ad-finite part of the quantum group U_q^{fin} and its integral form in U from Definition 2.4.19. Let $\lambda \in T_P^R$. It follows from Lemma 2.4.17 that the natural map $U^{\mathrm{fin}} \to \mathcal{M}_\lambda$ is in fact surjective. Now let $J_\lambda = \mathrm{Ann}_{U^{\mathrm{fin}}}(\mathcal{M}_\lambda)$ and let $U^\lambda = U^{\mathrm{fin}}/J_\lambda$. The map $U^{\mathrm{fin}} \to \mathcal{M}_\lambda$ now factors through a surjection $U^\lambda \to \mathcal{M}_\lambda$.

We now recall the construction of a map $f_\lambda : U^\lambda \to \Gamma(\mathcal{D}^\lambda)$ from the proof of [13, Proposition 4.8]. The map $U^\lambda \to \mathcal{M}_\lambda$ is a surjection of integrable $U^{\mathrm{res}}(\mathfrak{b})$-modules and so gives rise to a map

$$p^*(U^\lambda) \to p^*(\mathcal{M}_\lambda) = \mathcal{D}^\lambda$$

After taking global sections we obtain a map

$$\Gamma(p^*(U^\lambda)) \to \Gamma(\mathcal{D}^\lambda). \tag{5.3}$$

But since U^λ is in fact an integrable U^{res}-module, it follows from the tensor identity (e.g. [13, Lemma 3.13(a)]) that $U^\lambda \cong \mathrm{Ind}(U^\lambda) = \Gamma(p^*(U^\lambda))$ and hence we obtain from (5.3) an algebra homomorphism

$$f_\lambda : U^\lambda \to \Gamma(\mathcal{D}^\lambda)^{\mathrm{op}}.$$

By [13, Proposition 4.8], f_λ is injective since it becomes an isomorphism after tensoring with L. A completely similar construction gives rise to maps

$$f_{\lambda,n} : U^\lambda/\pi^n U^\lambda \to \Gamma(\mathcal{D}^\lambda/\pi^n\mathcal{D}^\lambda)^{\mathrm{op}}$$

for every $n \geq 1$. In fact there are natural maps

$$\Gamma(\mathcal{D}^\lambda)/\pi^n\Gamma(\mathcal{D}^\lambda) \to \Gamma(\mathcal{D}^\lambda/\pi^n\mathcal{D}^\lambda)$$

for every $n \geq 1$ and the maps $f_{\lambda,n}$ are obtained by postcomposing these with the reduction of f_λ modulo π^n. Taking inverse limits, one obtains by Proposition 5.2.4 a map $\widehat{U^\lambda} \to \Gamma(\widehat{\mathcal{D}^\lambda})^{\mathrm{op}}$. After extending scalars to L, this yields a map

$$\widehat{f_\lambda} : \widehat{U_q^\lambda} \to D^{\mathrm{op}}$$

where $\widehat{U_q^\lambda} := \widehat{U_L^\lambda}$.

Conjecture 5.4.13. *The map $\widehat{f_\lambda}$ is an isomorphism.*

In order to prove this conjecture, it would suffice to show that the reduction modulo π of f_λ is an isomorphism, by a straightforward inductive proof using the Five Lemma. Alternatively, we have the following:

Conjecture 5.4.14. *The ring U^{fin} is Noetherian.*

It is known that U_q^{fin} is Noetherian, see [48, Proposition 6.5]. However the proof was not written with integral forms in mind and it is unclear whether these arguments can be replicated for integral forms. We are not able to prove this conjecture for the moment. However we show how it implies Conjecture 5.4.13.

Lemma 5.4.15. *If Conjecture 5.4.14 holds, then for each $i \geq 0$, $R^i\Gamma(\mathcal{D}^\lambda)$ is a finitely generated U^{fin}-module.*

Proof. This follows from the conjecture by applying [12, Lemma 4.1.3] to $\mathcal{M}_\lambda$. □

Proposition 5.4.16. *Suppose $\lambda \in T_P^R$ is dominant and assume that Conjecture 5.4.14 holds. Then $\mathcal{D}^\lambda$ is acyclic.*

Proof. By [13, Theorem 4.12] and Proposition 4.3.10, the cohomology groups $R^i\Gamma(\mathcal{D}^\lambda)$ are π-torsion for $i \geq 1$. By the above Proposition, they are in fact bounded torsion, say there is $a \geq 0$ such that $\pi^a R^i\Gamma(\mathcal{D}^\lambda) = 0$ for all $i \geq 1$. But by Proposition 4.4.4 and Proposition 2.5.18(iii), we have that $R^i\Gamma(\mathcal{D}^\lambda) \otimes_R k = 0$ i.e. $R^i\Gamma(\mathcal{D}^\lambda) = \pi R^i\Gamma(\mathcal{D}^\lambda) = \cdots = \pi^a R^i\Gamma(\mathcal{D}^\lambda) = 0$. □

We now use these results to compute the global sections of $\widehat{\mathcal{D}_q^\lambda}$. We begin with a lemma:

Lemma 5.4.17. *If $M \in \mathscr{C}_R$ is Γ-acyclic and π-torsion free then*

$$\Gamma(M/\pi^n M) \cong \Gamma(M)/\pi^n\Gamma(M)$$

for all $n \geq 1$, and therefore $\Gamma(\widehat{M}) \cong \widehat{\Gamma(M)}$.

Proof. We have a short exact sequence

$$0 \to M \xrightarrow{\pi^n} M \to M/\pi^n M \to 0$$

in $\mathscr{C}_R$. Since M is Γ-acyclic this gives an exact sequence

$$0 \to \Gamma(M) \xrightarrow{\pi^n} \Gamma(M) \to \Gamma(M/\pi^n M) \to 0$$

as required. The last part now follows immediately from Proposition 5.2.4. □

We can now prove that Conjecture 5.4.13 follows from Conjecture 5.4.14.

Theorem 5.4.18. *Assume that U^{fin} is Noetherian and that λ is dominant. Then there is an isomorphism $\widehat{U_q^\lambda} := \widehat{U_L^\lambda} \to D = \Gamma(\widehat{\mathcal{D}_q^\lambda})$. If λ is furthermore regular then the functor Γ of global sections and the localisation functor Loc_λ are quasi-inverse equivalences of categories between the category $\mathrm{coh}(\widehat{\mathcal{D}^\lambda_{B_q}(G_q)})$ of coherent B_q-equivariant $\widehat{\mathcal{D}_q}$-modules on the analytic quantum flag variety and the category of finitely generated $\widehat{U_q^\lambda}$-modules.*

Proof. By the Lemma 5.4.17 and Proposition 5.4.16, we see that there is an isomorphism

$$\Gamma(\widehat{\mathcal{D}_q^\lambda}) \cong \widehat{\Gamma(\mathcal{D}^\lambda)_L}.$$

Moreover, by Lemma 5.4.15 the map $f_\lambda : U^\lambda \to \Gamma(\mathcal{D}^\lambda)$ is a map between finitely generated U^{fin}-modules, and it is injective. Hence we have a short exact sequence

$$0 \to U^\lambda \to \Gamma(\mathcal{D}^\lambda) \to C \to 0$$

of finitely generated U^{fin}-modules. Since C is torsion by [13, Proposition 4.8], it is in fact bounded torsion, say $\pi^n C = 0$. Now, since U^{fin} is assumed to be Noetherian and since $\pi \in U^{\mathrm{fin}}$ is central, the functor of taking π-adic completion is exact on finitely generated modules by [17, 3.2.3(ii)]. Thus we have a short exact sequence

$$0 \to \widehat{U^\lambda} \to \widehat{\Gamma(\mathcal{D}^\lambda)} \to \widehat{C} \to 0$$

where we still have $\pi^n\widehat{C} = 0$. Hence, after tensoring with L, we get an isomorphism

$$\widehat{U_q^\lambda} \to \widehat{\Gamma(\mathcal{D}^\lambda)_L}.$$

Putting the two maps together yields the isomorphism $\widehat{U_q^\lambda} \to D$. The equivalence of categories now follows from Theorem E. □

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